PhD Qualify Exam, PDE, Sep. 24, 2010

Show all works

E: easy, M: moderate, D: difficult

1.(E) Consider the wave equation $u_{tt} - c^2\Delta u = 0$, where $u = u(x,t)$, $x \in R^3$, and Δ is the Laplacian operator. For waves with spherical symmetry that is $u = u(\rho, t)$, where $\rho = |x|$. Derive the spherical symmetric solution for the wave equation in this special case and find its general solution. [10%]

2.(E) Consider the problem

$$\begin{cases} u_{tt} - c^2 u_{xx} + hu = F(x,t), & x \in R,\ t > 0, \\ u(x,0) = f(x), \quad u_t(x,0) = g(x), \end{cases} \tag{1}$$

where c and h are positive constants. Assuming that u, u_x , and u_t vanish as $x \longrightarrow \pm\infty$ for $t \geq 0$, and $\int_{-\infty}^{\infty} \left(u_t^2 + c^2 u_x^2 + hu^2\right) dx < \infty$ for $t \geq 0$. Show that the solution of the problem is unique. Hint: Use energy method. [10%]

3.(M) Let u be a nonnegative harmonic function in a ball $B_R(0)$. Show that for $|x| < R$, [10%]

$$\frac{R^{n-2}(R - |x|)}{(R + |x|)^{n-1}} u(0) \leq u(x) \leq \frac{R^{n-2}(R + |x|)}{(R - |x|)^{n-1}} u(0).$$

4.(E) Let $\Omega \subset R^n$ be open. Show that if there exists a function $u \in C^2(\overline{\Omega})$ vanishing on $\partial\Omega$ for which the quotient

$$\frac{\int_\Omega |\nabla u|^2}{\int_\Omega u^2}$$

reaches its infimum λ, then u is an eigenfunction for the eigenvalue λ, so that $\Delta u + \lambda u = 0$ in Ω. [10%]

5.(M) State a method of finding the Green's function for the eighth of a ball, [15%]

$$D = \{x^2 + y^2 + z^2 < a^2 : x > 0, y > 0, z > 0\}.$$

Write down the Green's function. Set up a Dirichlet problem which is related to the above Green's function. (No need to solve the Dirichlet problem.)

6.(M) Let Ω be an open bounded subset of R^n, with smooth boundary. Consider the boundary value problem

$$\begin{cases} -\Delta u + \lambda u &= f \quad \text{in } \Omega \\ \frac{\partial u}{\partial n} + \gamma u &= 0 \quad \text{on } \partial\Omega \end{cases} \tag{2}$$

where $\gamma > 0$.

(a) Give a weak formulation of this problem. [5%]

(b) Suppose that there is a weak solution for the problem (2), which is smooth in $\overline{\Omega}$. Show that this solution satisfies (2) in the usual (strong) sense. (Recall: We say that v is a weak derivative of u if $(v, \phi) = -(u, \phi')$ for all test function ϕ.) [10%]

7.(D) Find the solution for the problem [15%]

$$\begin{cases} u_t - u_{xx} = 0, & \\ u(x,0) = 0, & x > 0,\ t > 0, \\ u(0,t) = h(t), & \end{cases} \tag{3}$$

where $h(0) = 0$.

8.(M) Consider the Dirichlet problem for a circle [15%]

$$\begin{cases} u_{xx} + u_{yy} = 0, & x^2 + y^2 < a^2, \\ u(x,y) = h(x,y), & x^2 + y^2 = a^2, \end{cases} \tag{4}$$

where h is a continuous function on the circle. Assuming that the problem has the solution

$$u(\mathbf{x}) = \frac{a^2 - |\mathbf{x}|^2}{2\pi a} \int_{|\mathbf{x}'|=a} \frac{h(\phi)}{|\mathbf{x} - \mathbf{x}'|^2}\, a d\phi,$$

where $\mathbf{x} = (x,y) = (r\cos\theta, r\sin\theta)$ and $\mathbf{x}' = (x', y') = (a\cos\phi, a\sin\phi)$. Show that

$$\lim_{\mathbf{x}\to\mathbf{x_0}} u(\mathbf{x}) = h(\mathbf{x_0}),$$

for all $\mathbf{x_0}$ lies on the circle, that is $|\mathbf{x_0}| = a$. (Give a direct proof.)