Name:_ Algebra Exam 09/24/2010

Show ALL work for full credit.

(1) [20] (a) Prove that every group of order 159 is cyclic. (b) Let G be a group of order 153. Show that the center of G contains a group of order 9.

(2) [20] Let X and Y be non-zero $n \times n$ commuting matrices over $\mathbb{C}$. Prove that if the characteristic polynomial of X has no multiple roots, then the minimal polynomial of Y has no multiple roots.

(3) [20] Let $f : R \to S$ be a homomorphism between two commutative rings with identity. Let $p \subseteq S$ be a prime ideal. (a) Prove $f^{-1}(p)$ is a prime ideal. (b) Find all the prime ideals in $\mathbb{R}[x]$ and $\mathbb{C}[x]$. Part (a) defines a map $f^* : \mathbf{Spec}(S) \longrightarrow \mathbf{Spec}(R)$, from the set of prime ideals of S to the set of prime ideals of R. (c) Let $f : \mathbb{R}[x] \longrightarrow \mathbb{C}[x]$ be the natural map. Prove that f^* is either 2-to-1 or 1-to-1.

(4) [20] Let $L/\mathbb{Q}$ be a Galois extension of degree 15. (a) Show that every subextension $F/\mathbb{Q}$ with $\mathbb{Q} \subseteq F \subseteq L$ is normal. (b) Suppose that $f \in K[X]$ is irredcucible and its splitting field is L. Show that $\deg(f) = 15$.

(5) [20] Consider the noncommutative ring R generated by x and y over $\mathbb{C}$ with relation $yx - xy = 1$. Prove that R is simple, i.e. the only two-sided ideals of R are R and $\{0\}$.