Each question is 20 points.

1. Let $f_n, f_n' \in L^2[0,1]$ for each $n \in \mathbb{N}$. Suppose that $\{f_n\}$ is a sequence of absolutely continuous functions, and that there exist $f, g \in L^2[0,1]$ such that

$$\lim_{n\to\infty} \|f_n - f\|_{L^2[0,1]} = 0, \quad \lim_{n\to\infty} \|f_n' - g\|_{L^2[0,1]} = 0.$$

Show that there exists a number $c \in \mathbb{R}$ such that

$$f(x) = c + \int_0^x g(t)dt$$

for almost everywhere $x \in [0,1]$.

2. Let $\varphi \in L^1(\mathbb{R})$ be such that $\int_{\mathbb{R}} |\varphi(x)|dx = 1$. For $\varepsilon > 0$, define the function φ_ε by

$$\varphi_\varepsilon(x) = \varepsilon^{-1}\varphi(x/\varepsilon),$$

then for any function $f \in L^p(\mathbb{R})$, $1 \le p < \infty$ we have $\int_{\mathbb{R}} f(x-y)\varphi_\varepsilon(y)dy \to f$ in $L^p(\mathbb{R})$ as $\varepsilon \to 0$.

3. Let f, g be real-valued continuous functions defined on $\mathbb{R}$ and $g(x+1) = g(x)$. Show that

$$\lim_{n\to\infty} \int_0^1 f(x)g(nx)dx = (\int_0^1 f(x)dx)(\int_0^1 g(x)dx)$$

and

$$\lim_{n\to\infty} \int_0^{2\pi} f(x)\sin(nx)dx = 0.$$

4. Let f be a real valued function defined on $(0,\,1)$ such that $f(x) = 0$ if x is rational and $f(x) = \frac{1}{a}$ if x is irrational, where a is the first nonzero integer in the decimal representation of x. Prove that f is measurable and find $\int_{(0,\,1)} f dx$.

5. Show that $\int_0^\infty \frac{\sin t}{e^t - x} dt = \sum_{n=1}^\infty \frac{x^{n-1}}{n^2+1}$, $\forall\, x \in [-1,\,1]$.