Numerical Analysis 2008/9/25

1. (*15%*) Show that the Newton's method for finding a simple root of a function converges at least quadratically. (Recall that α is a simple root of f if $f(\alpha) = 0$ and $f'(\alpha) \neq 0$.)

2. (*15%*) Show that a second–order Runge–Kutta formula of the form

$$x(t+h) = x(t) + w_1 h f(t,x) + w_2 h f(t + \alpha h, x + \beta h f)$$

must impose these conditions:

$$\begin{aligned} w_1 + w_2 &= 1 \\ w_2 \alpha &= \tfrac{1}{2} \\ w_2 \beta &= \tfrac{1}{2} \end{aligned}$$

3. (*15%*) Find the constants A_0, A_1, A_2, x_0, x_1, and x_2 such that the Gaussian quadrature rule

$$\int_{-1}^{1} f(x)\,dx \approx A_0 f(x_0) + A_1 f(x_1) + A_2 f(x_2)$$

is exact for f in Π_5, which is the set of polynomials of degree less than or equal to 5. (Hint: the polynomial $x^3 - \frac{3}{5}x$ is orthorgonal to Π_2)

4. (*15%*) Prove that if a nonsingular matrix A has an LU-factorization in which L is a *unit* (i.e. $l_{ii} = 1$ for all i) lower triangular matrix, then L and U are unique.

5. (*10%*) Let $v^{(k)}, k = 1, 2, \ldots$, be the search directions in the method of descent. Prove that

$$v^{(k)} \perp v^{(k+1)}, \quad \text{for } k = 1, 2, \ldots$$

6. (*10%*) Write an efficient algorithm for evaluating

$$u = \sum_{i=1}^{n} \prod_{j=1}^{i} d_j$$

7. (*10%*) Determine the value of (a, b, c) that makes the function

$$f(x) = \begin{cases} x^3 & x \in [0,1] \\ \frac{1}{2}(x-1)^3 + a(x-1)^2 + b(x-1) + c & x \in [1,3] \end{cases}$$

a cubic spline.

8. (*10%*) Find the best approximation to $\sin(x)$ by a function $u(x) = \lambda x$ using the supremum norm.