PhD Qualify Exam, Analysis, Sept. 26, 2008

Show all works

1.[10%] Evaluate the integral $\int_{-\infty}^{\infty} e^{-x^2}\cos(xt)\,dx$.

2.[20%] Let $\mathbf{T}(x,y) = (e^x\cos y - 1, e^x \sin y) = (u,v)$ be a transformation: $R^2 \to R^2$, and f be a continuous function on R^2 with compact support. Let $J_{\mathbf{T}}$ be the Jacobian of $\mathbf{T}$. (**a**) Show that there are functions g_1 and g_2 from R^2 into R^1 such that $\mathbf{T}(x,y) = \mathbf{G_2} \circ \mathbf{G_1}(x,y)$, where $\mathbf{G_1}(x,y) = (g_1(x,y), y)$ and $\mathbf{G_2}(z,w) = (z, g_2(z,w))$. (**b**) Show that, for Riemann integral, $\int_{R^2} f(u,v)\,dudv = \int_{R^2} f(\mathbf{T}(x,y))\left|J_{\mathbf{T}}(x,y)\right| dxdy$. Use the result in part (a) to give a direct proof. (**c**) Under what conditions, does the formula in part (b) hold for Lebesgue integral?

3.[20%](Exam, Feb. 2006) Let C be the Cantor Set. (**a**) Show that $C + C = [0,2]$. Recall that $C+C \equiv \{x+y : x,y \in C\}$. (**b**) Compute the following quantities: (**i**) $\lambda_\alpha^\epsilon(C) \equiv \inf_{<B_i>} \sum_{i=1}^{\infty} r_i^\alpha$, where $< r_i >$ are radii of sequence of balls $< B_i >$ that covers C and for which $r_i < \epsilon$. (**ii**) $m_\alpha(C) \equiv \lim_{\epsilon\to 0} \lambda_\alpha^\epsilon(C)$. (**iii**) $\alpha_0 \equiv \sup\{\alpha : m_\alpha(C) = \infty\}$. ($\alpha_0$ is the Hausdorff dimension of C.) (**iv**) $m_{\alpha_0}(C)$. (Hausdorff measure of C.) (**v**) Find the Hausdorff measure of a unit disc in R^2.

4.[10%](Exam, Feb. 2007) Let $1 < p < \infty$, $f \in L^p(0,\infty)$, $F(x) = \frac{1}{x}\int_0^x f(t)\,dt$, and $0 < x < \infty$. (**a**) Prove that $\|F\|_p \le \frac{p}{p-1}\|f\|_p$. (**b**) Prove that the equality holds only if $f = 0$ $a.e.$ (**c**) What can you say about $p = 1$ and $p = \infty$?

5.[10%](Exam, Sept. 2004) Assume that $p > 0$ and $\int_E |f - f_k|^p dx \to 0$ as $k \to \infty$. Show that $\{f_k\}_{k=1}^\infty$ converges in measure on E to f.

6.[10%](Exam, Sept. 2004) Let ϕ be a nonnegative bounded measurable function on R^n such that $\phi(x) = 0$ for $|x| \ge 1$, and $\int \phi(x)\,dx = 1$. For $\epsilon > 0$, we define $\phi_\epsilon(x) = \epsilon^{-n}\phi(\frac{x}{\epsilon})$. If $f \in L^2(R^n)$, show that $\lim_{\epsilon\to 0} f * \phi_\epsilon = f$ in L^2. ($*$ means convolution.)

7.[20%](Exam, Feb. 2000) Let $\mathcal{M}$ be the collection of Lebesgue measurable subsets of R. μ be the Lebesgue measure on $(R,\mathcal{M})$, and μ_0 be the counting measure on $(R,\mathcal{M})$. Define ν on $(R,\mathcal{M})$ by $\nu(E) = \mu_0(E\cap\{0\}) - \mu(E\cap[0,1]) + \int_E \frac{1}{1+x^2}\,dx$. ($E \in \mathcal{M}$) (**a**) Find a Hahn decomposition of R for measure ν. (**b**) Find the Jordan decomposition of ν. (**c**) Find the Lebesgue decomposition of $|\nu|$ with respect to μ. (**d**) Compute the Radon-Nikodym derivative of the absolutely continuous part of $|\nu|$ with respect to μ.