PhD qualifying exam on Mathematical Programming September 28, 2007

1. (15 points) Consider the following Linear Programming problem (P):

$$\begin{aligned} \min \quad & x + 4y \\ \text{s.t.} \quad & 4x + y \geq 4 \\ & 2x + y \geq 3 \\ & x + 2y \geq 3 \\ & y \geq 0.5 \\ & 2x - y \leq 6 \\ & x - y \leq 2 \\ & x \geq 0, y \geq 0. \end{aligned}$$

   (a) (5 points) Draw the feasible domain on the $x - y$ plane and find out the optimal solution using the graphic method.

   (b) (10 points) Notice that $(1, 1)$ is not optimal. Transform (P) into the LP standard form and perform one iteration of the revised simplex method from the point $(1, 1)$. You have to compute the reduced cost vector, complete the minimum ratio test, and finally determine the pivoting variables.

2. (25 points) Prove or disprove (give counter examples to) the following statements:

   (a) (6 points) Consider a nonlinear program (P) which maximizes a function f subject to a feasible domain S. Suppose at $\bar{x} \in S$ the cone of feasible directions and the cone of ascent directions does not overlap, then there is no feasible direction along which f can be increased. Consequently, $\bar{x}$ must be a local maximum of (P).

   (b) (6 points) (Cont'd) Suppose f is differentiable at $\bar{x} \in S$ and $F_0 = \{d | \nabla f(\bar{x})^t d > 0\}$. If $\bar{x}$ is a local maximum of (P), F_0 can not overlap with the cone of feasible directions.

   (c) (6 points) The optimum for minimizing a concave function over a bounded polyhedral set P must be achieved at least on one of the extreme points of P.

   (d) (7 points) Consider the quadratic problem (QP)

$$\begin{aligned} \min \quad & \tfrac{1}{2}\mathbf{x}^t Q \mathbf{x} - \mathbf{f}^t \mathbf{x} \\ \text{s.t.} \quad & A\mathbf{x} = \mathbf{b} \end{aligned}$$

   where Q is a symmetric $n \times n$ matrix, $A \in R^{m \times n}$, $\mathbf{f}, \mathbf{x} \in R^n$, $\mathbf{b} \in R^m$. If problem (QP) has a local minimum point, then Q must be positive semi-definite on the null space of A.

3. (35 points) Consider the problem (P) that minimizes $f(x_1, x_2) = -5x_1 + 2x_2$ over the region $R = \{(x_1, x_2) | x_1 \geq x_2^3\}$.

(a) (5 points) Draw the picture of R.

(b) (7 points) Find the tangent cone and the related polar cone at $(0,0)$ for each region. (You may use the graph to represent the answer)

(c) (7 points) Determine all KKT points. Use the second order sufficient condition to determine which KKT point attains the minimum of (P).

(d) (8 points) Formulate and compute the (Lagrange) dual problem (D) of (P).

(e) (8 points) Verify the weak duality and/or the strong duality between (P) and (D).

4. (15 points)

(a) (8 points) State the line search with the Armijo's rule.

(b) (7 points) What is the advantage by using the Armijo's rule?

5. (10 points) Let

$$f(x) = \frac{1}{p}|x|^p, \quad p > 1, \; x \in R^n$$

Compute the conjugate function f^* and verify that $f^{**} = f$.