1. (15 points) Let $f \in C^1(\mathbb{R}^n)$ and suppose that for each open ball B that there exists a solution of the boundary value problem

$$\begin{aligned} -\Delta u = f \quad &\text{in } B \\ \partial u/\partial n = 0 \quad &\text{on } \partial B, \end{aligned}$$

where n is the outward unit normal vector field to ∂B. Show that $f \equiv 0$.

2. (15 points) Solve the nonlinear equation

$$u_t + uu_x = 0$$

with the initial condition $u(x,0) = x$. Sketch some of the characteristic lines.

3. (15 points) For $x \in \mathbb{R}$, $t \geq 0$, solve the problem

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= \cos x \\ u(x,0) &= \sin x \\ u_t(x,0) &= 1 + x \end{aligned}$$

4. (25 points) Let $L > 0$ and consider the following initial-boundary value problem:

$$\begin{cases} u_t - u_{xx} = 0, & 0 < x < L,\ 0 < t, \\ u(x,0) = f(x), & 0 \leq x \leq L, \\ u(0,t) = 0,\ u(L,t) = 1; & 0 \leq t, \end{cases}$$

where f is a continuous with piecewise continuous derivative on $[0,L]$, $f(0) = 0$, and $f(L) = 1$.

(a) Find the solution $u(x,t)$ of the problem.

(b) Is the solution C^∞ for $t > 0$? Justify your answer.

(c) Find $\lim\limits_{t \to \infty} u(x,t)$.

5. (15 points) Let $U \subset \mathbb{R}^n$ be a bounded, open set with a smooth boundary ∂U. Let $T > 0$ and set $U_T = U \times (0,T]$, $\Gamma_T = \overline{U}_T \setminus U_T$. Consider the following initial-boundary value problem:

$$\begin{cases} u_{tt} - \Delta u = f & \text{in } U_T \\ u = g & \text{on } \Gamma_T \\ u_t = h & \text{on } U \times \{t = 0\}. \end{cases}$$

Show that there exists at most one function $u \in C^2(\overline{U}_T)$ solving the above problem.

6. (15 points) Let λ, $a \in \mathbb{R}$, with $a > 0$. Let $u(x,y)$ be an infinitely differentiable function defined on a neighborhood of $D = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ such that

$$\begin{aligned} \Delta u + \lambda u = 0 \quad &\text{in } x^2 + y^2 < 1 \\ \partial u/\partial n = -au \quad &\text{on } x^2 + y^2 = 1. \end{aligned}$$

Here n denotes the outward unit normal vector field to ∂D.
Prove that if u is not identically zero in $x^2 + y^2 < 1$, then $\lambda > 0$.