Analysis, Sep., 2007

E: easy, M: moderate, D: difficult.

1. (E, 15 points) Let $f(x)$ be a real-valued function whose domain is measurable. Please show that the following statements are equivalent.

   (i) For any $\alpha \in R$, $\{x : f(x) > \alpha\}$ is measurable.

   (ii) For any $\alpha \in R$, $\{x : f(x) \geq \alpha\}$ is measurable.

2. (M, 15 points) Suppose that $f \in L^1(R)$. Let $F(x) = \int_R f(t)\frac{\sin xt}{t}dt$.

   (a) Prove that F is differentiable on R and find $F'(x)$.

   (b) Determine whether F is absolutely continuous on every compact subinterval of R.

3. (D, 20 points) $1 < p < \infty$, $f \in L^p(0, \infty)$. Let $F(x) = \frac{1}{x}\int_0^x f(t)dt$, $0 < x < \infty$.

   (a) Prove that
$$\|F\|_p \leq \frac{p}{p-1}\|f\|_p.$$

   (b) Prove that the equality holds only if $f = 0$ a.e.

4. (M, 15 points) Let $\{f_n\}$ be a sequence of non-negative measurable functions on $(-\infty, \infty)$ such that $f_n \to f$ a.e., and suppose that $\int_R f_n \to \int_R f < \infty$. Please prove that for each measurable set E,
$$\int_E f_n \to \int_E f.$$

5. (E, 10 points) Please prove that
$$\lim_{A\to\infty}\int_0^A \frac{\sin x}{x}dx = \frac{\pi}{2}.$$
   (Hint: $\frac{1}{x} = \int_0^\infty e^{-xt}dt, \quad x > 0$.)

6. (E, 15 points) A step function is, by definition, a finite linear combination of characteristic functions of bounded intervals in R^1. Let $f \in L^1(R^1)$. Prove that there is a sequence $\{g_n\}$ of step functions so that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} |f(x) - g_n(x)| dx = 0.$$

7. (M, 10 points) Let g be a non-negative measurable function on $[0, 1]$. Please show that

$$\log \int g(t)dt \geq \int \log(g(t))dt$$

whenever the right side is defined.