1. (10 points) Consider the following Linear Programming problem (P):

$$\begin{aligned}
\min \quad & Dx_4 + x_5 + Ex_6 + Fx_7 \\
\text{s.t.} \quad & x_2 + Gx_4 + x_5 + 3x_7 = H \\
& x_3 - 2x_4 + 2x_5 + Cx_6 - x_7 = 2 \\
& x_1 - x_5 + 2x_6 + x_7 = 3 \\
& x_i \geq 0, \quad \text{for } i = 1, 2, \cdots, 7.
\end{aligned}$$

where C, D, E, F, G, H are real numbers. Please specify a complete range of values for numbers C, D, E, F, G, H so that the following assertions are valid. Let us denote $B = \{x_2, x_3, x_1\}$

   (a) (2.5 points) The first constraint makes (P) inconsistent (have no feasible solution).

   (b) (2.5 points) B is an optimal basis, but is not unique.

   (c) (2.5 points) B is a feasible basis, but (P) is unbounded.

   (d) (2.5 points) B is a feasible basis, x_6 is a candidate for entering B, and x_3 is the one to leave B.

2. (15 points) Consider the following three types of mathematical program:

$$\textbf{(P)} \quad \text{minimize} \quad h(x) + \frac{f(x)}{g(x)} \quad \text{subject to} \quad x \in S$$

$$\begin{aligned}
\textbf{(Q)} \quad \min_r \quad & q(r) = h(x(r)) + \frac{f(x(r))}{r} \\
\text{where} \quad & x(r) = \text{argmin}\{h(x) + \frac{f(x)}{r} \mid g(x) = r, x \in S\}
\end{aligned}$$

$$\begin{aligned}
\textbf{(R)} \quad \min_r \quad & q(r) = h(x(r)) + \frac{f(x(r))}{r} \\
\text{where} \quad & x(r) = \text{argmin}\{h(x) + \frac{f(x)}{r} \mid g(x) \geq r, x \in S\}
\end{aligned}$$

Here, S is a compact convex set in $\mathbb{R}^n$, $f, g, h : S \mapsto \mathbb{R}$ are such that f and h are convex, g is concave, and $f(x) \geq 0$ and $g(x) > 0$. For each $r > 0$,

   (a) (10 points) If we set $q(r) = \infty$ for $r > \max\{g(x) | x \in S\}$, show that the three problems **(P)**,**(Q)**, and **(R)** are equivalent.

(b) (5 points) Which of the three problems is easier to solve than the other two (write reasons)? Design an algorithm of your own to solve that problem.

3. (25 points) Prove or disprove the following statements:

   (a) (5 points) Let P be a polyhedral set in R^n. Assume that $P \neq \phi$ and $P \neq R^n$. Then, $\bar{x}$ is an extreme point of P if and only if $P \backslash \{\bar{x}\}$ is a convex set.

   (b) (5 points) The optimum for maximizing a convex function over a bounded polyhedral set P must be achieved at least on one of the extreme points of P.

   (c) (5 points) Consider the quadratic problem

$$\begin{aligned} \min \quad & \tfrac{1}{2}\mathbf{x}^t Q\mathbf{x} - \mathbf{f}^t\mathbf{x} \\ \text{s.t.} \quad & A\mathbf{x} = \mathbf{b} \end{aligned}$$

   where Q is a symmetric $n \times n$ matrix, $A \in R^{m\times n}$, $\mathbf{f}, \mathbf{x} \in R^n$, $\mathbf{b} \in R^m$. If $\mathbf{x}^*$ is a local minimum point, then it must be a global minimum point.

   (d) (10 points) Given the following two linear programs:

$$(P1) \quad \begin{aligned} \min \quad & (\mathbf{c})^T\mathbf{x} \\ \text{s.t.} \quad & A\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq 0 \end{aligned} \qquad\qquad (P2) \quad \begin{aligned} \min \quad & (\mathbf{c}')^T\mathbf{x} \\ \text{s.t.} \quad & A\mathbf{x} = \mathbf{b}' \\ & \mathbf{x} \geq 0 \end{aligned}$$

   where $A \in R^{m\times n}$, $\mathbf{c}, \mathbf{c}', \mathbf{x} \in R^n$, $\mathbf{b}, \mathbf{b}' \in R^m$, $\mathbf{c}' = \beta\mathbf{c}$, $\mathbf{b}' = \lambda\mathbf{b}$, $\lambda > 0$ and $\beta \in R$. Assume that $(P1)$ has at least two feasible solutions but has a unique finite optimum. Moreover, $(P1)$ is nondegenerate.

      i. (4 points) $(P2)$ may be degenerate.

      ii. (3 points) $(P2)$ may be unbounded.

      iii. (3 points) $(P2)$ may have multiple optimal solutions.

4. (30 points) Consider maximizing $f(x_1, x_2) = -5x_1 + 2x_2$ over the following three regions. Call problem (I) as to maximize f over $R_1 = \{(x_1, x_2) | x_1 \geq x_2^3\}$; problem (II) as to maximize f over $R_2 = \{(x_1, x_2) | (x_2^2 - x_1)(x_2^2 - 2x_1) \leq 0\}$; and problem (III) as to maximize f over $R_3 = \{(x_1, x_2) | -4x_1 + x_2^2 \leq 0; -2x_1 + x_2 \leq 0; x_1 \geq 0\}$.

   (a) (3 points) Draw the picture of R_1, R_2 and R_3.

   (b) (3 points) Find the tangent cone and the related polar cone at $(0, 0)$ for each region. (You may use the graph to represent the answer)

   (c) (4 points) Find the cone of feasible directions at $(0, 0)$ for each region.

   (d) (4 points) Determine all KKT points of Problem (I). Use the second order sufficient condition to determine which KKT point attains the maximum of Problem (I).

(e) (4 points) Formulate and compute the (Lagrange) dual problem of Problem (III).

(f) (4 points) Let S be the statement " If the cone of feasible directions and the cone of ascent directions at the same point $\bar{x}$ does not overlap, then $\bar{x}$ is a local maximum." In which problem ((I),(II) or (III)) is S a *false* statement? Why?

(g) (4 points) In which problem ((I),(II) or (III)) is S a *true* statement? Use this problem to explain the *Farkas* lemma.

(h) (4 points) In order to make the statement S in (d) a *true* statement, what kind of condition is needed?

5. (10 points) This problem is related to one the the most important method in the class of Quasi-Newton methods.

   (a) (5 points) State the *Davidon-Fletcher-Power (DFP)* method.

   (b) (5 points) What is the advantage of the (DFP) method when comparing with the Newton's method and the steepest descent method?

6. (10 points) Let

$$f(x) = \log(e^{x_1} + e^{x_2} + \cdots + e^{x_n})$$

where $x = (x_1, x_2, \ldots, x_n)$. Compute the conjugate function f^* and verify that $f^{**} = f$.