Partial Differential Equation

E: easy, M: moderate, D: difficult.

1. (E, 10 points) Let $u(x,y)$ be a nonconstant harmonic function in the disk $x^2+y^2<R^2$. Define for each $0<r<R$,

$$M(r)=\max_{x^2+y^2=r^2} u(x,y).$$

Prove that $M(r)$ is a monotone increasing function in the interval $(0,R)$.

2. (E, 15 points) Let $u(r,\theta)$ be a harmonic function in the disk

$$D=\{(r,\theta)|0\le r<R,-\pi<\theta\le\pi\},$$

such that u is continuous in the closed disk $\bar{D}$ and satisfies

$$u(R,\theta)=\begin{cases}\sin^2 2\theta, & |\theta|\le\pi/2,\\ 0, & \pi/2<|\theta|\le\pi.\end{cases}$$

(a) Evaluate $u(0,0)$.

(b) Show that $0<u(r,\theta)<1$ holds at each point (r,θ) in the disk.

3. (M, 20 points) Use the energy method to prove uniqueness for the problem

$$u_{tt}-c^2u_{xx}+hu=F(x,t),\quad -\infty<x<\infty, t>0.$$

$$\lim_{x\to\pm\infty}u(x,t)=\lim_{x\to\pm\infty}u_x(x,t)=\lim_{x\to\pm\infty}u_t(x,t)=0,\quad t\ge 0.$$

$$\int_{-\infty}^{\infty}(u_t^2+c^2u_x^2+hu^2)dx<\infty,\quad t\ge 0,$$

$$u(x,0)=f(x),\quad u_t(x,0)=g(x),\quad -\infty<x<\infty,$$

where c and h are positive constants.

4. (M, 15 points) Consider the Cauchy problem

$$u_{tt}-c^2u_{xx}=0,\quad -\infty<x<\infty, t>0,$$

$$u(x,0)=f(x),\quad u_t(x,0)=g(x),\quad -\infty<x<\infty.$$

Fix $T>0$. Please prove that the above problem in the domain $-\infty<x<\infty$, $0\le t\le T$ is well-posed for $f\in C^2(R)$, $g\in C^1(R)$.

5. (M, 10 points) Let $D_R \equiv R^2 \setminus B_R$ be the exterior of the disk with radius R centered at the origin. Find the Green function (for the Laplace operator) of D_R.

6. (M, 20 points) Solve the following heat problem:

$$\begin{aligned} u_t - ku_{xx} &= A\cos\alpha t, \quad 0 < x < 1, \ t > 0, \\ u_x(0,t) = u_x(1,t) &= 0, \quad t \geq 0, \\ u(x,0) &= 1 + \cos^2 \pi x, \quad 0 \leq x \leq 1. \end{aligned}$$

7. (M, 10 points)

(a) Find the eigenfunction expansion of the function on $[0, 2]$

$$f(x) = \begin{cases} x, & 0 \leq x \leq 1, \\ 1, & 1 \leq x \leq 2 \end{cases}$$

with respect to the (classical Fourier) orthonormal system

$$\left\{\sqrt{\frac{1}{2}}\right\} \cup \{\cos n\pi x\}_{n=1}^{\infty} \cup \{\sin n\pi x\}_{n=1}^{\infty}$$

(b) Does the series you obtain in (a) converge to f ?