國立成功大學應用數學所 數值分析 博士班資格考 **March, 9, 2007**

1. Since

$$(\sqrt{2}-1)^6 = (3-2\sqrt{2})^3 = 99-70\sqrt{2}$$
$$= \frac{1}{(\sqrt{2}+1)^6} = \frac{1}{(3+2\sqrt{2})^3} = \frac{1}{99+70\sqrt{2}},$$

please point out which one formula gives a minimal round-off error and explain why? (15%)

2. Consider the initial value problem

$$\text{(I.V.P.)}\begin{cases} y' = f(t,y), & a \le t \le b, \\ y(a) = \alpha. \end{cases}$$

Show that the difference method

$$w_0 = \alpha,$$
$$w_{i+1} = w_i + a_1 f(t_i, w_i) + a_2 f\left(t_i + \beta, w_i + \delta f(t_i, w_i)\right), \; i = 0, 1, \ldots, n-1,$$

cannot give a 3^{rd}-order local truncation error, i.e. $O(h^3)$ where $h = \frac{b-a}{n}$, for any choice of constants a_1, a_2, β and δ. (15%)

3. Consider a 4-digit decimal system. Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4+\varepsilon \\ 1 & 2+\varepsilon \end{bmatrix}$ where $\varepsilon = 10^{-2}$.

   (a) Show that $\text{rank}(A) = 2$. (5%)

   (b) Show that for a given $b \in \mathbb{R}^3$ the least square problem,

   $$\min_{x\in\mathbb{R}^2} \|Ax - b\|_2, \qquad (LS)$$

   can not be usually solved by using the normal equation. (10%)

   (c) Find $A^\dagger$, denotes the pseudo-inverse (generalized inverse) of A. Let $b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, use $A^\dagger$ to construct a solution of problem (LS) so that the constructed solution has at least 3 significants. (10%)
   (Hint: $f\ell(1.0+\varepsilon^2) = 1.0$, where $f\ell(\cdot)$ is the floating operator.)

4. Definition: A sequence $\{p_n\}_{n=1}^{\infty}$ is said to be convergent to p of order α with asymptotic error constant λ if $\lim_{n\to\infty} \frac{|p_{n+1}-p|}{|p_n-p|^{\alpha}} = \lambda$.

   (a) Let $g : [a, b] \longrightarrow [a, b]$ be a continuous function. Show that there is a point $p^* \in [a, b]$ such that $g(p^*) = p^*$. (5%)

   (b) Let $p_{n+1} = g(p_n)$ (with a given p_0) defined a fixed point iteration. Please give a sufficient condition such that the fixed point iteration is convergent of order k, where k is a positive integer. (10%)

   (c) Show that the Newton's iteration is a local quadratic method (i.e. $\alpha = 2$), whenever the iteration is convergent. (10%)

5. Calculate $30^{\frac{1}{3}}$ upto 3 digits after the decimal point and estimate the error bound of your answer. (10%)

6. Let $x_0, x_1, \ldots, x_n \in \mathbb{R}$ be $n+1$ distinct numbers and $y_0, y_1, \ldots, y_n \in \mathbb{R}$. Show that there is a unique polynomial $P_n(x)$ of degree n such that $P_n(x_i) = y_i$, for $i = 0, 1, \ldots, n$. (10%)