Functional Analysis

96.3.9

1. Let T be a compact operator on a Hilbert space H, and let A be the unique positive square root of T^*T.
   (a) Show that $\|Ah\| = \|Th\|$ for all h in H. (10%)
   (b) Show that there is a unique operator U such that $\|Uh\| = \|h\|$ where $h \perp \ker T$, $Uh = 0$ where $h \in \ker T$, and $UA = T$. (10%)

2. Let X be a Banach space and suppose $\{x_n\}$ is a sequence in X such that for each x in X there are unique scalars $\{\alpha_n\}$ such that $\lim_{n\to\infty} \|x - \sum_{k=1}^{n} \alpha_k x_k\| = 0$. Such a sequence is called a Schauder basis.
   (a) Prove that X is separable. (5%)
   (b) Let $Y = \Big\{\{\alpha_n\} : \{\alpha_n\}$ is a sequence of scalars such that $\sum_{n=1}^{\infty} \alpha_n x_n$ converges in $X\Big\}$ and for $y = \{\alpha_n\}$ in Y define $\|y\| = \sup_n \|\sum_{k=1}^{n} \alpha_k x_k\|$. Show that Y is a Banach space. (10%)
   (c) Show that there is a bounded bijection $T : X \to Y$. (10%)
   (d) If $n \geq 1$ and f_n is a linear functional on X defined by $f_n\big(\sum_{k=1}^{\infty} \alpha_k x_k\big) = \alpha_n$, show that $f_n \in X^*$. (10%)

3. Suppose X is an infinite-dimensional normed space. If $S = \{x \in X : \|x\| = 1\}$, then the weak closure of S is $\{x \in X : \|x\| \leq 1\}$. (15%)

4. If X is compact, $k \in C(X \times X)$, and μ is a regular Borel measued on X, show that

$$Kf(x) = \int k(x,y) f(y)\, d\mu(y)$$

   defines a compact operator on $C(X)$. (15%)

5. Suppose f is a complex continuous function in $\mathbb{R}^n$, with compact support. Prove that $\psi P_j \to f$ uniformly on $\mathbb{R}^n$, for some $\psi \in \mathcal{D}$ and for some sequence $\{P_j\}$ of polynomials. (15%)