There are seven problems in the exam. Work out all seven of them.

[15%] 1. For what positive integers n is it true that the only abelian groups of order n is cyclic. Show your arguments.

[15%] 2. Let G be a group and let $H_1, \ldots, H_n$ be subgroups of G of finite index. Set $H = H_1 \cap H_2 \cap \cdots \cap H_n$. Show that H has finite index in G and

$$[G : H] \leq [G : H_1][G : H_2] \cdots [G : H_n].$$

[15%] 3. Let R be a commutative ring with unity and let M be an ideal of R. Show that M is maximal if and only if R/M is a field. (An ideal M of R is said to be maximal if J is an ideal of R containing M, then $J = M$ or $J = R$.)

[15%] 4. Describe all ring homomorphisms of $\mathbb{Z} \oplus \mathbb{Z}$ into $\mathbb{Z}$. (Remember that the identities may not be preserved by a homomorphism.)

[15%] 5. Let F be a field. Let I be an ideal of $F[x]$ such that $p(x)q(x) \in I$ implies $p(x) \in I$ or $q(x) \in I$. Prove that I is a maximal ideal in $F[x]$.

[15%] 6. Let F be a finite field of order p^n where p is a prime and n a positive integer. Show that there is exactly one subfield of p^m elements for each divisor m of n.

[10%] 7. Let R be a ring and A an R-module. Prove that if $f : A \to A$ is an R-homomorphism such that $f \circ f = f$, then $A = \mathrm{Ker} f \oplus \mathrm{Im} f$.