PhD Qualify Exam, Analysis, Feb. 24, 2006

Show all works

1.[10%] Explain the meaning of the Lebesgue integral $\int_R f(x)\,dx$. Begin by defining Lebesgue outer measure, measurable sets and measurable functions. Then explain how the Lebesguge integral is defined.

2.[15%] Suppose $f \in L^1(R)$. Let $F(x) = \int_R f(t)\frac{\sin xt}{t}\,dt$.

a. Prove that F is differentiable on R and find $F'(x)$.

b. Determine whether or not F is absolutely continuous on every compact subinterval of R.

3.[15%] Suppose $f_n \in L^2[0,1], n = 1, 2, 3, \cdots$, and $\sum_1^\infty \|f_n\|_2 < \infty$. Prove that

a. $\sum_1^\infty |f_n(x)| < \infty$ *a.e.*

b. If $f(x) = \sum_1^\infty f_n(x)$ *a.e.*, then $f \in L^2[0,1]$ and $\|f\|_2 \leq \sum_1^\infty \|f_n\|_2$.

4.[10%] Let g be a nonnegative measurable function on $[0,1]$. Then

$$\log \int g(t)\,dt \geq \int \log(g(t))\,dt$$

whenever the right side is defined.

5.[10%] Let $< A_n >_{n\in N}$ be a sequence of connected subsets of the topological space X such that $A_n \cap A_{n+1} \neq \emptyset$ for all $n \in N$. Prove that the set $\cup_{n\in N} A_n$ is connected.

6.[15%] Let (X, B, μ) be a finite measure space. Suppose that ν is a measure defined on B such that ν is absolutely continuous with respect to μ and such that $\nu(X)$ is finite. Let g be the Radon-Nikodym defivative of ν with respect to μ. Prove that $\int_X f d\nu = \int_X fg d\mu$.

7.[15%] Let $f \in L^2[0,1]$ and suppose that for each q such that $1 < q < \infty$, $\left| \int_0^1 fg \right| \leq \|g\|_q$ for every $g \in C[0,1]$.

a. Prove that $f \in \cap_{1<p<\infty} L^p[0,1]$.

b. Is $f \in L^\infty[0,1]$? Explain your answer.

8.[10%] Find the Hausdorff dimension of $C \times C$, where C is the Cantor Set, by computing the following quantities:

First

$$\lambda_\alpha^\epsilon(C \times C) = \inf \sum_{i=1}^{\infty} r_i^\alpha,$$

where $< r_i >$ are radii of sequence of balls $< B_i >$ that covers $C \times C$ and for which $r_i < \epsilon$.

Second,

$$m_\alpha(C \times C) = \lim_{\epsilon \to 0} \lambda_\alpha^\epsilon(C \times C).$$

Finally, Hausdorff dimension of $C \times C$ is $\inf\{\alpha : m_\alpha(C \times C) = \infty\}$.