Algebra Qualifying Examination Feb 2006

Answer all the problems and show all your works.

1. (15%) Let G be the group of all 5×5 invertible matrices. Let
$$x = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$
and $H = \langle x, y \rangle$ the subgroup generated by x and y. Determine the order of H.

2. (15%) Let G is a group of order 231. Show that the Sylow 11-subgroup of G is in the center of G.

3. (10%) Let R be a commutative ring with identity such that every nonzero ideal is generated by finitely many elements. Show that if $\{I_i\}_{i=1}^{\infty}$ is any sequence of ideals in R satisfying $I_i \subseteq I_{i+1}$ for all i, then there is an integer n such that $I_i = I_n$ for all $i \geq n$.

4. (15%) Let R be a ring with identity. Let A be a right R-module, B a R-bimodule and C a left R-module. Show that there is an isomorphism of abelian groups:
$$A \otimes_R (B \otimes_R C) \cong (A \otimes_R B) \otimes_R C.$$

5. (15%) Let F be a field and let V be a finite dimensional vector space over F. Let $\phi : V \times V \longrightarrow F$ be a bilinear map satisfying $\phi(x, x) = 0$ for all $x \in V$. Assume that for any nonzero element $x \in V$, there is an element $y \in V$ such that $\phi(x, y) \neq 0$. Show that $\dim V$ is even.

6. (15%) Let E be a finite Galois extension over F and $F \subset K \subset E$ an intermediate field. Show that K is normal over F if and only if $Gal(E/K)$ is a normal subgroup of $Gal(E/F)$, where $Gal(E/K)$ and $Gal(E/F)$ denote the group of all K-automorphisms and F-automorphisms of E, respectively. (K is said to be normal over F if every nonzero irreducible polynomial of $F[x]$ which has a root in K splits into linear factors in $K[x]$.)

7. (15%) Let K be a subfield of a field F. For any subset A of F, let $K(A)$ denote the smallest subfield of the field F containing K and A. Let S be a subset of F. Suppose that u is algebraic over $K(S)$ but u is not algebraic over $K(S - \{v\})$ for some $v \in S$. Show that v is algebraic over $K((S - \{v\}) \cup \{u\})$.

THE END