博士班資格考 Functional Analysis

September 26, 2005

1. [20%] Let H be a complex Hilbert space and A be a linear functional on H. Define $ker(A) = \{y \in H | A(y) = 0\}$.

(a) Prove that if A is continuous, then $ker(A)$ is a closed subspace of H.

(b) Prove that if A is discontinuous, then $ker(A)$ is dense in H.

2. [10%] Let A be a compact normal operator on a complex Hilbert space H. (Recall that the normal means that $AA^* = A^*A$.) Prove that there is an orthonormal basis for H made up of eigenvectors for A.

3. [10%] Let A be a bounded linear operator on a Hilbert space with $\|A\| < 1$. Prove that $(I - A)^{-1}$ exists (here I is the identity operator).

4. [10%] Show that there exists a bounded linear functional λ on $L^\infty([-1,1])$ such that $\lambda(f) = f(0)$ for each $f \in C([-1,1])$. (The space $L^\infty([-1,1])$ is defined using the Lebesgue measure.)

5. [20%] Let X be a Banach space and $P : X \to X$ be a (not necessarily continuous) linear operator with domain X such that $P^2 = P$. Let R be the range of P and N be the kernel of P.

(a) Prove that X is the direct sum of N and R.

(b) Prove that P is continuous if and only if both R and N are closed.

6. [10%] Let X be a Banach space and $B : X \times X \to R$ be a bilinear form such that

(i) for each fixed $x \in X$, the map $y \to B(x, y)$ is continuous, and

(ii) for each fixed $y \in X$, the map $x \to B(x, y)$ is continuous.

Show that there exists a bounded linear operator $T : X \to X^*$ such that $B(x, y) = \langle Tx, y\rangle$, where $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{R}$ is the usual duality form: $\langle f, x\rangle = f(x)$. Conclude that there exists a constant $C > 0$ such that $B(x, y) \leq C\|x\| \, \|y\|$.

7. [10%] Let X be a Banach space and T be a bounded linear mapping of X into itself satisfying

$$\|x\| \leq K\|Tx\|$$

for all $x \in X$, for some $K \in \mathbb{R}$. Prove that the range of T is closed.

8. [10%] Prove that a bounded sequence in a separable, reflexive Banach space contains a weakly convergent subsequence.