泛函分析

博士班資格考

94.2.25

PM1:00-4:00

1. If $1 < p < \infty$, prove that the unit ball of $L^p(R^n)$ is strictly convex; this means that if

$$\|f\|_p = \|g\|_p = 1, \quad f \neq g, \quad h = \frac{1}{2}(f+g),$$

then $\|h\|_p < 1$. Show that this fails in $L^1(R^n)$, and in $L^\infty(R^n)$.

2. Prove that $W^{1,2}(R^n)$ is a Hilbert space. (Recall : (*))

3. Use the Fourier transform method to show that the equation

$$u_{xx} + u_{yy} + u_{zz} = -q(x,y,z),$$

where q is a continuous, positive function on R^3, has the formal solution

$$u(r) = \frac{1}{4\pi}\int_{R^3} \frac{q(s)}{\|s-r\|}ds, \quad r = (x,y,z).$$

4. Suppose that X and Y are Banach spaces, and suppose that Λ is a linear mapping of X into Y, with the following property:
For every sequence x_n in X for which $x = \lim x_n$, and $y = \lim \Lambda x_n$ exist, it is true that $y = \Lambda x$.
Prove that Λ is continuous.

5. Suppose that X is an infinite-dimensional normed linear space and $S = \{x \in X : \|x\| = 1\}$. Show that

   (a) the weak closure of S is $\{x \in X : \|x\| \leq 1\}$;

   (b) $\{x \in X : \|x\| \leq 1\}$ is not compact.

(*): v is called the α-th weak derivative of u and is denoted by $D^\alpha u$ if it satisfies

$$\int_{R^n} \varphi v \, dx = (-1)^{|\alpha|}\int_{R^n} u \, D^\alpha \varphi \, dx, \text{ for all } \varphi \in C_0^{|\alpha|}(R^n)$$

$W^{1,2}(R^n) \equiv \{u : D^\alpha u$ exist and $D^\alpha u \in L^2(R^n)$ for all $|\alpha| \leq 1\}$.