PhD Qualifying Exam, Analysis

2005.9.23

1. Let $\{f_n\}$ be a sequence of functions in $L^p[0,1]$, $1 < p < \infty$, which converge almost everywhere to a function f in $L^p[0,1]$, and suppose that there is a constant M such that $\|f_n\| \leq M$ for all n.

   (a) (10%) For each function g in $L^q[0,1]$ and $\frac{1}{p} + \frac{1}{q} = 1$, show that
$$\int_0^1 fg = \lim_{n\to\infty} \int_0^1 f_n g.$$

   (b) (5%) Prove or disprove the result for $p = 1$.

2. Let f, g be real-valued continuous functions defined on R and $g(x+1) = g(x)$.

   (a) (10%) Show that
$$\lim_{n\to\infty} \int_0^1 f(x)g(nx)dx = (\int_0^1 f(x)dx)(\int_0^1 g(x)dx).$$

   (b) (5%) State and use (a) to prove the Riemann-Lebesgue lemma.

3. (10%) Compute the limit
$$\lim_{n\to\infty} \int_0^\infty \frac{n\cos x}{1+n^2x^{\frac{3}{2}}}dx.$$

4. Let $f : [0,1] \to \mathbb{R}$ be defined by
$$f(x) = \begin{cases} \frac{1}{q}, & \text{if } x = \frac{p}{q}, \text{ where } p, q \in \mathbb{N} \text{ and have no common factors.} \\ 0, & \text{if x is irrational.} \end{cases}$$

   (a) (10%) Prove that f is continuous ar every irrational point of $[0,1]$ and discontinuous elsewhere.

   (b) (10%) Prove that, in spite of having infinitely many discontinuities, f is Riemann integrable on $[0,1]$ and $\int_0^1 f(x)dx = 0$.

5. Let f be of bounded variation on $[a,b]$ and $V(x)$ be its total variation on $[a,x]$, $x < b$.

   (a) (10%) Show that
$$V'(x) = |f'(x)|.$$

   (b) (5%) Use (a) to show that
$$\int_a^b |f'(x)|dx \leq V[a,b],$$

   (c) (10%) Show that if the equality holds in the above inequality, then f is absolutely continuous on $[a,b]$.

6. (a) (10%) Let g be a nonnegative measurable function on $[0,1]$. Show that
$$\exp(\int \log(g(t))dt) \leq \int g(t)dt.$$

   (b) (5%) Explain the inequality as the arithmetic mean greater than the geometric mean.