Qualified Examination: Partial Differentiation Equation

Sep., 2005

Do all problems. (E: easy, M: moderate, D:difficult)

1. (15 points) (M)
   Let $u(x) = u(x_1, \cdots, x_n) \in C^2$ for $|x| < a$; $u \in C^0$ for $|x| \le a$. And $u \ge 0$, $\Delta u = 0$ for $|x| < a$. Show that for $|\xi| < a$,

$$\frac{a^{n-2}(a - |\xi|)}{(a + |\xi|)^{n-1}} u(0) \le u(\xi) \le \frac{a^{n-2}(a + |\xi|)}{(a - |\xi|)^{n-1}} u(0).$$

2. (15 points) (M)
   Let $\Omega \subset R^n$ be open. Show that if there exists a function $u \in C^2(\bar{\Omega})$ vanishing on $\partial\Omega$ for which the quotient

$$\frac{\int_\Omega |\nabla u|^2}{\int_\Omega u^2} = \min\{\frac{\int_\Omega |\nabla w|^2}{\int_\Omega w^2} : w \in C^2(\Omega), w = 0 \quad on \quad \partial\Omega, w \not\equiv 0\} = \lambda,$$

   then $\Delta u + \lambda u = 0$ in Ω. That is, λ is an eigenvalue of $-\Delta$ and $u(x)$ is its eigenfunction.

3. (15 points) (M)
   Let $u(x,t)$ be a solution of $u_{tt} - \Delta u = 0, \quad t \in R^+, \quad x \in R^3$. Suppose that $a > 0$ and that $u(x, 0) = u_t(x, 0) = 0$ for $|x| \ge a$.

   (a) Show that $u(x,t) = 0$ in the double cone $|x| \le |t| - a$ for $|t| \ge a$.

   (b) Show that there is a constant $C > 0$ such that

$$\int_{R^3} u^2(x,t)dx \le C, \quad \text{for all } t > 0.$$

4. (15 points) (E)
   Solve the initial boundary value problem

$$u_t + cu_x = -\lambda u, \quad x, t > 0.$$

$$u(x, 0) = 0, \quad x > 0; \quad u(0,t) = g(t), t > 0.$$

   (Please discuss all the possible situations.)

5. (20 points) (M)
Use the Fourier transform method to solve the initial value problem

$$\begin{aligned} u_t &= u_{xx}, \quad -\infty < x < \infty, \quad t > 0, \\ u(x,0) &= f(x), \quad -\infty < x < \infty. \end{aligned}$$

And prove that u satisfies the following inequality

$$\|u\|_p(t) \leq \frac{1}{(4\pi t)^{\frac{1}{2}(\frac{1}{q}-\frac{1}{p})}}\|f\|_q, \quad t > 0,$$

for $1 \leq q \leq p \leq \infty$. (Note that the L^p, L^q norms are with respect to x.)

6. (20 points) (M)
Let $Lu = \sum_{k=1}^{3} a_k(x)\frac{\partial u}{\partial x_k}$, $x = (x_1, x_2, x_3) \in \Omega$, where Ω is an open set in R^3 and $a_k(x) \in C^\infty(\Omega)$.
Given $f \in L^2(\Omega)$, we say that u is an L^2 weak solution of $Lu = f$ in Ω if $u \in L^2_{loc}(\Omega)$ and

$$< u, L'\psi > = < f, \psi >, \quad \forall \psi \in C_c^\infty(\Omega),$$

where $L'u = -\sum_{k=1}^{3} \frac{\partial(a_k u)}{\partial x_k}$.

Suppose that there is a constant c such that

$$< f, \phi > \leq c\|L'\phi\|_{L^2(\Omega)}, \quad \forall \phi \in C_c^\infty(\Omega).$$

Please prove that there exists an L^2 weak solution of

$$Lu = f.$$

(Note: $< f, g > = \int_\Omega fg dx$, $\|f\|_{L^2} = (\int_\Omega f^2 dx)^{\frac{1}{2}}$.)