Master Qualifying Exam, Analysis

2005.9.23

1. (20%) State the monotone convergence theorem, dominated convergence theorem, Fatou's lemma, Funini's theorem.

2. (10%) Show that the set of rational numbers on $[0, 1]$ is of measure 0 and give an example which is an uncountable measure zero set.

3. (10%) Give an example that a sequence of functions is convergent in measure but not pointwise convergent.

4. (10%) Show that the Cantor-Lebesgue function is of bounded variation but not absolute continuous.

5. (10%) Compute the limit

$$\lim_{n\to\infty}\int_1^\infty \frac{n\cos x}{1+n^2x^{\frac{3}{2}}}dx.$$

6. (a) (10%) Let g be a nonnegative measurable function on $[0, 1]$. Show that

$$\exp(\int \log(g(t))dt) \le \int g(t)dt.$$

   (b) (5%) Explain the inequality as the arithmetic mean greater than the geometric mean.

7. (10%) Let $\{f_n\}$ be a sequence of functions in $L^p[0,1]$, $1 < p < \infty$, which converge almost everywhere to a function f in $L^p[0,1]$, and suppose that there is a constant M such that $\|f_n\| \le M$ for all n. For each function g in $L^q[0,1]$ and $\frac{1}{p}+\frac{1}{q}=1$, show that

$$\int_0^1 fg = \lim_{n\to\infty}\int_0^1 f_n g.$$

8. Let f, g be real-valued continuous functions defined on R and $g(x+1) = g(x)$.

   (a) (10%) Show that

$$\lim_{n\to\infty}\int_0^1 f(x)g(nx)dx = (\int_0^1 f(x)dx)(\int_0^1 g(x)dx).$$

   (b) (5%) Use this result to show that

$$\lim_{n\to\infty}\int_0^{2\pi} f(x)\sin(nx)dx = 0$$