機率論

博士班資格考 93.2.20

PM 2:00 — 5:00

1. Let X be a random variable with finite second moment, and μ be a median of X. Prove $|EX - \mu| \le \sqrt{2\mathrm{Var}X}$ (10%)

2. Let X, Y be independent random variables with common $N(0,1)$ distribution. Prove $X+Y$ and $X-Y$ are independent. (10%)

3. Let X, Y be iid random variables which have finite vatiance and if $Z_1 = X+Y$, $Z_2 = X-Y$ are independent, prove that X, Y, Z_1, Z_2 are normal distributed. (10%)

4. Let X, Y be independent random variables. Prove $X+Y$ is normal distributed if and only X, Y are both normal. (10%)

5. Prove that if $X_1, X_2, \ldots, X$ is a martingale, then for every $\varepsilon > 0$,

$$P\{\sup_n |X_n| > \varepsilon\} \le \frac{1}{\varepsilon}\int_{\{\sup_n |X_n| > \varepsilon\}} |X| dP \le \frac{1}{\varepsilon} E|X|.$$ (10%)

6. For sequence of r.v's $\{X_n\}$, if $\lim_{n\to\infty} ES_n = 0$ where $S_n = \sum_{i=1}^{n} X_i$. Prove (20%)
   (i) $\dfrac{S_n - ES_n}{n} \to 0$ in probability.
   (ii) $\dfrac{S_n - ES_n}{n}$ not necessarily converges almost surely.

7. Suppose $\{X_n\}$ is a sequence of iid symmetric r.v's (20%)
   (a) Show that for a sequcnec $\{a_n\} \subset \mathbb{R}$, $\sum_{k=1}^{\infty} a_k X_k$ converge almost surely as $n \to \infty$, then $\sum_{k=1}^{\infty} a_k^2 < \infty$.
   (b) Give an example to show that $\sum_{k=1}^{\infty} a_k^2 < \infty$ is not sufficient. (Hint: use $a_n = \dfrac{1}{n^{\alpha}}$ and X_1 with $E\left[|X_1|^{\frac{1}{\alpha}}\right] = +\infty$)

8. Suppose that $\{Z_n\}$ denotes the n coin tossing results for a fair coin. Show that there is a constant c, independent of n and t so that $P\{\max_{k\le n} \dfrac{Z_k}{\sqrt{n}} \ge t\} \le \dfrac{c}{t^2}$. (10%)