博士班資格考 Analysis

Friday, September 17, 2004

1. [10%] Let f be a real-valued function defined on a set E of $\mathbb{R}^n$. Then f is called a *Lebesgue measurable function* on E, or simply a *measurable function*, if for every finite a, the set $\{x \in E : f(x) > a\}$ is a measurable subset of $\mathbb{R}^n$. Prove that f is measurable if and only if for every open set G in $\mathbb{R}^1$, the inverse image $f^{-1}(G)$ is a measurable subset of $\mathbb{R}^n$.

2. [10%] Suppose that $\{f_k\}$ is a sequence of measurable functions which converges almost everywhere in a set E of finite measure to a finite limit f. Given ε, $\eta > 0$, show that there is a closed subset F of E and an integer K such that $|E - F| < \eta$ and $|f(x) - f_k(x)| < \varepsilon$ for $x \in F$ and $k > K$.

3. [10%] Let f and f_k, $k = 1, 2, \ldots$, be measurable and finite a.e. in E. Prove that if $f_k \to f$ a.e. on E and $|E| < \infty$, then $\{f_k\}$ converges in measure on E to f, i.e. for every $\varepsilon > 0$, $\lim_{k\to\infty} |\{x \in E : |f(x) - f_k(x)| > \varepsilon\}| = 0$.

4. [10%] If $p > 0$ and $\int_E |f - f_k|^p \to 0$ as $k \to \infty$, show that $\{f_k\}$ converges in measure on E to f.

5. [10%] Let $\{f_k\}$ be a sequence of nonnegative measurable functions defined on E. If $f_k \to f$ and $f_k \le f$ a.e. on E, show that $\int_E f_k \to \int_E f$ as $k \to \infty$.

6. [10%] Let $1 < p < \infty$, $f \in L^p(\mathbb{R}^n)$ and $g \in L^1(\mathbb{R}^n)$. Prove that $f * g \in L^p(\mathbb{R}^n)$, and $\|f * g\|_p \le \|f\|_p \|g\|_1$, where $(f * g)(x) = \int_{\mathbb{R}^n} f(t)g(x - t)dt$.

7. [10%] For $f \in L(\mathbb{R}^1)$, define the *Fourier transform* $\hat{f}$ of f by $\hat{f}(x) = \int_{-\infty}^{\infty} f(t)e^{-ixt}\,dt$. Show that if f and g belong to $L(\mathbb{R}^1)$, then $\widehat{(f * g)}(x) = \hat{f}(x)\hat{g}(x)$.

8. [10%] Let $\phi(x)$, $x \in \mathbb{R}^n$, be a bounded measurable function such that $\phi(x) = 0$ for $|x| \ge 1$ and $\int \phi = 1$. For $\varepsilon > 0$, let $\phi_\varepsilon(x) = \varepsilon^{-n}\phi(x/\varepsilon)$. If $f \in L(\mathbb{R}^n)$, show that $\lim_{\varepsilon\to 0}(f * \phi_\varepsilon)(x) = f(x)$ a.e. in $\mathbb{R}^n$.

9. [10%] If $f \in L^2(0, 2\pi)$, show that

$$\lim_{k\to\infty} \int_0^{2\pi} f(x)\cos kx\,dx = \lim_{k\to\infty} \int_0^{2\pi} f(x)\sin kx\,dx = 0.$$

10. [10%] A sequence $\{f_k\}$ in L^p is said to *converge weakly* to a function f in L^p if $\int f_k g \to \int fg$ for all $g \in L^{p'}$, where p' satisfies that $\frac{1}{p} + \frac{1}{p'} = 1$. Prove that if $f_k \to f$ in L^p norm, $1 < p < \infty$, then $\{f_k\}$ converges weakly to f in L^p.