機率論

(博士班)

92.9.19
AM 9:00 – 12:00

1. Let $(\Omega, \mathcal{A}, P)$ be a probability space. X and Y are integrable random variables. Assume that $\mathcal{B}$ is a sub-σ-field of $\mathcal{A}$ such that X is $\mathcal{B}$-measurable
   (a) Show that $E(Y\,|\,\mathcal{B}) = X$ implies $E(Y\,|\,X) = X$.
   (b) Show by a counterexample that $E(Y\,|\,X) = X$ does not imply that $E(Y\,|\,\mathcal{B}) = X$.
   (c) If $E(X\,|\,Y) = Y$ and $E(Y\,|\,X) = X$, show that $X = Y$ a.s.

2. Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d. r.v.'s $P(X_n = 1) = P(X_n = -1) = \dfrac{1}{2}$ compute the limiting distribution as $n \to +\infty$ of

$$Y_n = \frac{X_1 + 2X_2 + 3X_3 + \cdots + nX_n}{\sqrt{1 + 4 + 9 + \cdots + n^2}}.$$

3. Let $(X_n, \mathcal{F}_n)_{n\geq 1}$ be a submartingale. If $a, b \in \mathbb{R}$ with $a < b$, let $U_n(a,b)$ be the number of upcrossings of (a,b) by the sequence $X_1, \ldots, X_n$. Show that for each $n \geq 1$

$$E[U_n(a,b)] \leq \frac{E[(X_n - a)^+]}{b - a}$$

4. Let $\{Z_{ni}, n \geq 0, i \geq 1\}$ be i.i.d., nonnegative integer-valued r.v.'s with $0 < m = E[Z_{ni}] < \infty$. Let $Y_0 = 1$ and for each n, put

$$Y_{n+1} = 0, \qquad \text{if } Y_n = 0$$
$$Y_{n+1} = \sum_{i=1}^{Y_n} Z_{ni}, \qquad \text{if } Y_n > 0$$

Describe the limiting behavior of $X_n = \dfrac{Y_n}{m^n}$ by the martingale theroy for the case of $m < 1$.