PhD Qualify Exam, PDE, Sep. 19, 2003

Show all works

1. Suppose that $g \in C^1(R)$ and that $M \equiv \sup_R |g(x)| < \infty$. Consider the initial value problem

$$\begin{cases} u_t + u^2 u_x = 0 & \text{for } x \in R,\ t > 0 \\ u(x,0) = g(x) & \text{for } x \in R. \end{cases} \tag{1}$$

   (a) Show that a C^1 solution $u(x,t)$ satisfies $|u(x,t)| \le M$ for as long as the solution exists. [5%]

   (b) Suppose that at some $x_0 \in R$, $g(x_0)g'(x_0) < 0$. Show that a C^1 solution breaks down in finite time. (Hint: study the behavior of u_x along characteristics.) [5%]

2. Let u be a solution of the wave equation in all of $R^3 \times R$. Suppose that $a > 0$ and that $u(x,0) = u_t(x,0) = 0$ for $|x| \ge a$.

   (a) Show that $u(x,t) = 0$ in the double cone $|x| \le |t| - a$ for $|t| \ge a$. [5%]

   (b) Show that there is a constant $C > 0$ such that

$$\int_{R^3} u^2(x,t)\,dx \le C, \quad \text{for all } t > 0.$$

   (Hint: Show that there is a finite energy solution of $w_{tt} - \Delta w = 0$ such that $w_t = u$.) [5%]

3. Let u be a nonnegative harmonic function in a ball $B_R(0)$. Show that for $|x| < R$, [10%]

$$\frac{R^{n-2}(R-|x|)}{(R+|x|)^{n-1}}u(0) \le u(x) \le \frac{R^{n-2}(R+|x|)}{(R-|x|)^{n-1}}u(0).$$

4. Classify all the linear transformations T under which the Laplacian is invariant, [10%]

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}.$$

5. Suppose that Ω is an open bounded subset of R^n, with smooth boundary. Let $u(x,t)$ be a smooth solution of the problem

$$\begin{cases} u_t - \Delta u + c(x,t)u = 0, & \text{in } \Omega \times (0,\infty), \\ u = 0, & \text{on } \partial\Omega \times [0,\infty), \\ u(x,0) = g(x), & \text{in } \Omega. \end{cases} \tag{2}$$

(a) Suppose that $g(x) \geq 0$ and $|c(x,t)| \leq K$, for all $(x,t) \in \overline{\Omega} \times [0,\infty)$. Show that $u(x,t) \geq 0$ for all $(x,t) \in \overline{\Omega} \times [0,\infty)$. [10%]

(b) Can you deduce that $|u(x,t)|$ remains bounded as $t \to \infty$? [10%]

6. (a) Use the method of characteristics to solve

$$\begin{cases} u_y + (u_x)^4 = 0, \\ \quad u(x,0) = \frac{3}{4}x^{\frac{4}{3}}. \end{cases} \tag{3}$$

What is the domain of existence of the solution? [10%]

(b) Is it possible to apply the Local Existence Theorem to the problem

$$\begin{cases} u_y + (u_x)^4 = 0, \\ \quad u(x,x) = x, \quad \text{for} \quad x > 0, \\ \quad u_x(x,x) = \frac{1}{\sqrt[3]{4x}} \quad \text{for} \quad x > 0. \end{cases} \tag{4}$$

Justify your answer. [10%]

7. Let $u, v \in C^1(\overline{\Omega})$ be conjugate harmonic functions, i.e., $u_x = v_y$ and $u_y = -v_x$, in a simply connected domain Ω with C^1 boundary in R^2. Show that on the boundary curve $\partial\Omega$,

$$\frac{du}{dn} = \frac{dv}{ds}, \quad \frac{dv}{dn} = -\frac{du}{ds},$$

where $\frac{d}{dn}$ denotes differentiation in the direction of the outer normal and $\frac{d}{ds}$ differentiation in the counter-clockwise tangential direction. Show that these relations can be used to reduce the Neumann problem for u to the Dirichlet problem for v. [10%]

8. Let $\Omega \subset R^n$ be open. Show that if there exists a function $u \in C^2(\overline{\Omega})$ vanishing on $\partial\Omega$ for which the quotient

$$\frac{\int_\Omega |\nabla u|^2}{\int_\Omega u^2}$$

reaches its infimum λ, then u is an eigenfunction for the eigenvalue λ, so that $\Delta u + \lambda u = 0$ in Ω. [10%]