Math. Prog. Ph.D. Qualifying Exam.

91.9.20

1. Consider the following nonlinear programming problem:

(28%)

$$\begin{aligned} \max \quad & f(x) = x_1^2 + (x_2 - 2)^2 \\ \text{s.t.} \quad & g_1(x) = (x_2 - 2) + (x_1 - 1)^3 \le 0 \\ & g_2(x) = (x_1 - 1)^3 - (x_2 - 2) \le 0 \\ & g_3(x) = -x_1 \le 0 \\ & g_4(x) = -x_2 \le 0 \end{aligned}$$

(a) Draw the graph of the feasible domain and find out all global optimal solutions.

(b) Draw the normal cones at (0,3) and (0,2), then draw the gradients $\nabla f(0,3)$ and $\nabla f(0,2)$. Explain the K-K-T conditions graphically at (0,3) and (0,2).

(c) Find out the Lagrange Multipliers associated with (0,3) and (0,2) respectively.

(d) Are the K-K-T conditions necessary or sufficient to the optimality? Does every global optimal solution in (a) satisfy the K-K-T conditions? Why?

2. Given the primal problem (18%)

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad & g(x) \le 0 \\ & x \in \mathbf{X} \subset \mathbf{R^n} \end{aligned}$$

where $g : \mathbf{R^n} \to \mathbf{R^m}$, X= compact, $f, g \in C^1$

(a) Show that $h(\lambda) = \inf\{f(x) + \lambda^T g(x) | x \in X\}$ is concave on $\mathbf{R^m}$.

(b) Define $x(\lambda) = \{y \in \mathbf{R^n} \mid y \text{ minimizes } f(x) + \lambda^T g(x) \text{ over } X\}$. Show that, if $\bar{x} \in x(\bar{\lambda})$, then $g(\bar{x})$ is a subgradient of h at $\bar{\lambda}$.

(c) In a linear case, if problem (P) is

$$\begin{aligned} \min \quad & c^t x \\ \text{s.t.} \quad & A(x) \ge b \\ & x \ge 0, \end{aligned}$$

prove that the Lagrangian dual is just the dual problem of (P) in the sense of Linear Program.

3. Let X be a non-empty set in $\mathbf{R^n}$ and $f : \mathbf{R^n} \to \mathbf{R}$. Consider the conjugate function f^* defined as follows: (18%)

$$f^*(u) = \sup\{(x,u) - f\{x\} : x \in \mathbf{X}\}$$

(a) Interpret f^* geometrically.

(b) Let $f(x) = e^x$, $x \in \mathbf{R}$ and $g(x) = \frac{1}{p}|x|^p$, $p > 1$, $x \in \mathbf{R}$. Compute $f^*(u)$ and $g^*(u)$

(c) In what case, $f = f^*$

4. This problem relates to algorithms. Let

A == steepest descent method

B = Newton's method

C == Conjugate gradient method

D = Barrier function method. E == penalty method

F = Kelley's cutting plane method.

Answer the folowing questions with appropreate reasons. (18%)

(a) Which methods involve the line search?

(b) Which methods require first-order informations of the objective function?

(c) why can a cutting plane method be treated as a dual method?

5. Let $a_1 < a_2 < \ldots < a_m$ be m real numbers and $t_1, t_2, \ldots t_m$ be positive.
Define f on **R** by (18%)

$$f(x) = \sum_{j=1}^{m} t_j \,| x - a_j |$$

(a) Show that $f(x)$ is convex.

(b) Show that

$$\partial f(x) = \begin{cases} \sum\{j: a_j < x\} t_j - \sum\{j: a_j > x\} t_j, & if \quad x \notin \{a_1, a_2, \ldots a_m\} \\ \sum\{j: a_j < x\} t_j - \sum\{j: a_j > x\} t_j + [-t_{j0}, t_{j0}], & if \quad x = a_{j0} \end{cases}$$

(c) Design an algorithm to minimize f.