**91 年度博士班資格考，科目：數值分析** 題目共七題有三頁，總分：110

1. (i) Show that the nontrivial fixed point of the equation $9x = x^3, x \in [0, \infty]$ is unstable, that is you cannot find the nontrivial fixed point by using the fixed point iteration $9x_{k+1} = x_k^3$ with arbitrary $x_0 \neq 3 \in [0, \infty]$. 10%

   (i) Please develop a new fixed iteration method such that the non-trivial fixed point of Problem 1 .(i) is stable, 10%

**2.** (i) Develop a numerical method to compute the transcendental number $\pi$ if there is no internal trigonometric function in your calculator. 10%

   (i) Give a brief error analysis of the method you proposed, 10%

**3.** Let $f \in C^3(a,b)$ and $|f'''(x)| < M$ for $x \in (a,b)$. Consider a centered difference formula to be an approximation of *f''(x),* i.e.,

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h} \quad , \quad x \in (a,b)$$

Show that the centered difference formula is numerical unstable, i.e., the error function $e_x$,(h) which is defined by the difference of the derivative *f'(x)* and the approximation formula satisfies

$$e_x(h) \leq \frac{\varepsilon}{h} + \frac{h^2}{6} M \, ,$$

where $\varepsilon = max\{|f(x+h) - f\lambda(f(x+h))|, |f(x-h) - f\lambda(f(x-h))|\}$. 10%

4. Show that there is a unique quadratic function pa satisfying the conditions

$$p_2(0) == a_0, \quad p2(l) = a_1 \quad and \quad \int_0^1 p_2(x)dx = \bar{a}$$

with given $a_0, a_1$ and $\bar{a}$. 10%

5. Consider the nonlinear integral equation

$$u(t) = \int_0^1 k(t, s, u(s))ds$$

over the space $U = C[0, 1]$. Assume $K \in \mathbf{C}([0, 1] \times [0, 1] \times IR)$ and is continuously differentiable with respect to its third argument. Introducing an operator $F : U \rightarrow U$ through the formula

$$F(u)(t) = u(t) - \int_0^1 k(t, s, u(s))ds \, , t \in [0,1]$$

the integral equation can be written in form $F(u) = 0$.

   (i) Describe a Newton-type method to solve the nonlinear integral equation, 10%

   (i) Explore sufficient conditions for the convergence of the Newton-type method,10%

**6.** Is it possible to use $af(x + h) + bf(x) + cf(x — h)$ with suitably chosen coefficients *a, b, c* to approximate *f'''(x)?* How many function values are needed to approximate *f'''(x)?* 10%

*7.* (i) Describe the prototype projection method for solving the linear system $Ax = b$, where $A$ *is* an $n \times n$ matrix and $b$ is an $n$ vector. 10%

(ii) Let $x$ be the approximate solution obtained from a projection process onto $K$ and orthogonal to $L == AK$.. Define $\tilde{r} = b - A\tilde{x}$ . Show that $\|\tilde{r}\|_2 \leq \|r_0\|_2$ ,where $r_0 = b - Ax_0$ and $x_0$ is an initial guess for the projection method. *5%*

(iii) Let $A$ be symmetric positive definite and $L == K$. Show that $\|\tilde{d}\|_A < \|d_0\|_A$ , where $\tilde{d} = x_* - \tilde{x}$ , $d_0 == x_* - x_0$ ,and $x_* = A^{-1}b$.
Here $\|d\|_A = \sqrt{d^T Ad}$. *(5%)*