Qualified Examination: Functional Analysis

Feb 21.2003

Do all problems. (E: easy. M: moderare. D:difficult).

1. (15) (M) Prove that the space $L^1(R)$ is not reflexive.

2. (20) (E) Consider the operator A defined by

$$Af(x)=\int K(x,y)f(y)dy \text{ ----(*)}$$

from $L^r(R^n)$ into $L^p(R^n)$. where $\frac{1}{p}+\frac{1}{r}=1$,$1\le p\le\infty$

(a) Prove that if $K(x,y)\in L^p(R^n\times R^n)$, then A is a bounded operator.

(b) Let X, Y be bounded closed sets in R^n. Denote by μ the Lebesgue measure. Prove that if $K(x, y)$ is continuous on $X \times Y$, then the operator A defined by (*) is a compact operator from $L^r(Y, \mu)$ into $L^p(X, \mu)$

3. (15) (M) Let Y be a finite dimensional linear subspace of a normed space X. Show that Y must be closed.

4. (20) (E) Let X be a normed linear space,and let X^* be its dual with the norm $||X^*||=sup\{|f(x)| :f\in X^* ,x\in X ,||x||\le 1\}.X,$).

(a) Prove that X^* is a Banach space.

(b) Prove that for each $x \in X$, the mapping f $\to$ $f(x)$ is a bounded linear functional on X^* with norm ||x||.

5. (15) (E) Let H be a separable Hilbert space, and let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis of H. $T : H \to H$ is a bounded operator with $\sum_{n=1}^{\infty}\| Te_n \|^2<\infty$ Prove that T is a compact operator.

6. (15) (D) Use the Fourier transform method to show that if $r = \{x, y, z\}$, then the equation

$$u_{xx}+u_{yy}+u_{zz}=-q(x,y,z)\,,$$

where q is a continuous, positive function on R^3 , has the formal solution

$$u(r)=\frac{1}{4\pi}\int_{R^3}\frac{q(s)}{\| s-r \|}ds\,.$$