<u>博士班</u> **91.09.20**

AM 9:00—12:00

Functional Analysis Qualified Examination (Fall, 2002)

1. (a) Suppose M is a closed subspace of a normed linear space X such that $\dim X/M < \infty$. Prove that $M + N$ is closed for every subspace N.
   (b) Give an example of two closed subspaces M and N of a normed linear space X such that $M + N$ is not closed.

2. Let X be compact and suppose that S is a Banach subspace of $C(X)$. If E is a closed subset of X such that for every g in $C(E)$ there is an f in S with $f|_E = g$, show that there is a constant c > 0 such that for each g in $C(E)$, there is an f in S with $f|_E = g$ and $\max\{|f(x)| : x \in X\} \le c\max\{|g(x)|: x \in E\}$.

3. Suppose X is an infinite-dimensional normed linear space and $S = \{x \in X : \|x\| = 1\}$. Show that
   (a) the weak closure of S is $\{x \in X : \|x\| \le 1\}$;
   (b) $\{x \in X : \|x\| \le 1\}$ is not compact.

4. Show that
   (a) In $L^p\{0,1\}$, $1 < p < \infty$, every x with $\|x\| = 1$ is an extreme point of the unit sphere $S = \{x : \|x\| \le 1\}$.
   (b) In $L^\infty[0,1]$ the extreme points of the unit sphere are those x such that $|x(t)| = 1$ a.e.
   (c) The unit sphere in $L^1[0, 1]$ has no extreme points.
   (d) $L^1[0,1]$ is not the dual of any normed linear space.

5. Let (X, Ω, μ) be a (r-finite measure space and for $\phi \in L^\infty(\mu)$ let

$M_\phi: L^2(\mu) \to L^2(\mu)$

be the multiplication operator defined as

$$M_\phi(f)(t) = \phi(t)f(t) \quad \text{for} \quad f \in L^2(\mu).$$

(a) Give necessary and sufficient conditions on (X, Ω, μ) and ϕ for M_ϕ

to be
compact.

(b) Find $\sigma(M_\phi)$ (the specturm of M_ϕ).