Algebra Ph.D. Qualifying Examination Feb 2003

Answer all the problems and show all your works.

1. (20%) Let G be a group of order 56 with no element of order 14. Prove that
   (i) the Sylow 7-subgroups of G are not normal in G, and
   (ii) the Sylow 2-subgroup of G is normal in G and is isomorphic to $Z_2 \times Z_2 \times Z_2$? where Z_2 is a group of order 2.

2. (15%)Let p be a prime.
   (i) Show that every group of order p^2 is abelian.
   (ii) Suppose that G is an non-abelian group of order p^3. Show that the center of G is riontrivial.

3. (10%) (i) Show that every group can be embedded into a symmetric group S_n for some n.
   (ii) Show that every group can be embedded into an alternating group A_n for some n.

4. (10%) Let D be a principal ideal domain. Show that I is prime ideal in D if and only if it is a maximal ideal.

5. (10%) Let G be a group. Show that End G is a ring if and only if G is abelian, where End G is the set of all homomorphisms from G to G and the addition and the multiplication on End G are defined as follows:

$$(f+g)(a) = f(a)g(a), \text{ and } \quad f \bullet g(a) = f(g(a)),$$

   for any $f, g \in$ End G and $a \in G$

6. (10%) Let F be a finite field. Show that the order of F is a power of a prime.

7. (10%) Let Q be the field of all rational numbers. Show that

$$Q(\sqrt{2}, \sqrt{3}) = Q(\sqrt{2} + \sqrt{3}).$$

8. (15%) Let F be a field and A, B and C F-vector spaces. Show that

$$(A \otimes B) \otimes C \cong A \otimes (B \otimes C)$$

   as F-vector spaces.

THE END