Algebra Qualifying Examination, September 2002

Answer all the problems and show all your works.

1. (15%) Let G be a nonabelian group of order 6. Show that G is isomorphic to S_3, the symmetry group of degree 3.

2. (15%) Let G be a group of order 56. Suppose that G has no element of order 14. Show that the Sylow 2-subgroup of G is normal in G.

3. (20%) Let G is a group of order 231. Show that the Sylow 11-subgroup of G is in the center of G.

4. (15%) Let R be a commutative ring with identity and
$$f(x) = a_0+a_1x+\ldots..+a_nx^n \in R\{x\}.$$
Show that $f(x)$ is a unit in $R[x]$ if and only if a_0 is a unit in R and $a_{1,\ldots,}a_n$ are nilpotent elements in R.

5. (10%)Let R be a integral domain and $a, b \in R$. Suppose $a^n= b^n$ and $a^m = b^m$, where m,n are positive integers and $(m,n) = 1$. Prove that $a = b$.

6. (10%)An integral domain D is called a Euclidean domain if there is a function $d: D\backslash\{0\} \to \mathbf{Z}^+$ such that
(1). $d(a) \leq d(ab)$ for any $a, b \in D\backslash\{0\}$ and
(2). for any $a \in D$ and $b \neq 0$, there are $q,r \in D$ such that $a = qb + r$, where $d(r) < d(b)$ or $r = 0$.
Show that $d(a) == d(e)$ if and only if a is a unit.

7. (15%) (i) Show that a finite extension E of F is also an algebraic extension of F.
(ii) Let K be a field and E an extension of K. Suppose $u, v \in E$ are roots of an irreducible polynomial $f(x)$ £$\in K[x]$. Show that there is a unique field isomorphism
$$\sigma : K(u)\text{-} K(v)$$
such that $\sigma |K = id_K$ and $\sigma (u) = v$.

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