GENARAL ANALYSIS

(PhD Program Qualify Exam: Feb. 23, 2001)

I. (15%) Given $a_n \in \{0,1,2,3,4\}$ and $b_n \in \{0,1,2,3,4,5,6\}$ and define the set P by

$$P = \Big\{(x,y) \in [0,1] \times [0,1] \,\big|\, x = \sum_{n=1}^{\infty} \frac{a_n}{5^n}, \quad y = \sum_{n=1}^{\infty} \frac{b_n}{7^n}$$
$$a_n \in \{0,2\}, b_n \in \{1,3,5\}\Big\}$$

What does P look like? Compute the Lebesgue measure of P. Is P open, closed, compact, perfect? Can you evaluate the Hausdorff dimension of P?

II. (15%) Given $x = (x_1, x_2, \cdots, x_n) \in \mathbf{R}^n$ with $|x| = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2}$. Show that the function $\rho(x)$ defined in $\mathbf{R}^n$ by $\rho(x) = \exp(\frac{1}{|x|^2-1})$ if $|x| < 1$ and $\rho(x) = 0$ if $|x| > 1$ belongs to $C_c(\mathbf{R}^n)$, the space of infinitely differentiable with compact support. Compute the integral $\int_{\mathbf{R}^n} \rho(x)dx$.

III. (15%) If the function $f(x)$ is absolutely continuous on $[a,b]$, then the length s of the curve $y = f(x)$ can be computed according to the formula

$$s = \int_a^b \sqrt{1 + |f'|^2(x)}dx$$

(You need to start from the definition of arc-length!)

IV. (20%) Let Ω be a bounded domain in $\mathbf{R}^n$, If u is a measurable function on Ω such that $|u|^p \in L^1(\Omega)$ for some $p \in \mathbf{R}$, we define

$$\Phi_p(u) \equiv \left[\frac{1}{|\Omega|}\int_\Omega |u|^p dx\right]^{1/p}$$

where $|\Omega|$ denotes the measure of Ω. Show that

(a) $\lim_{p\to\infty} \Phi_p(u) = \sup_\Omega |u|$;

(b) $\lim_{p\to-\infty} \Phi_p(u) = \inf_\Omega |u|$;

(c) $\lim_{p\to 0} \Phi_p(u) = \exp\left[\frac{1}{|\Omega|}\int_\Omega \log|u|dx\right]$.

(d) Φ is logarithmically convex in $1/p$, i.e. if $p \le q \le r$ and $1/q = \lambda/p + (1-\lambda)/r$ then $\log \Phi_q \le \lambda \log \Phi_p + (1-\lambda)\log \Phi_r$

V. (15%) Give $1 \le p < \infty$ and a sequence $\{f_n\}_{n=1}^{\infty} \subset L^p(\Omega)$ and $f \in L^p(\Omega)$.

(a) Can you define the weak convergence of f_n to f in $L^p(\Omega)$ (denoted by $f_n \overset{w}{\rightharpoonup} f$).

(b) For $1 < p < \infty$ show that $f_n \overset{w}{\rightharpoonup} f$ in $L^p(\Omega)$ if and only if

$$\sup_n \|f_n\|_{L^p(\Omega)} < \infty \qquad \text{and} \qquad \int_E f_n dx \to \int_E f dx$$

for all bounded measurable set $E \subset \Omega$. Is it true for $p = 1$?

VI. (20%) Let (X, μ) be a measure space, and let $1 \le p \le \infty$ and $C > 0$. Suppose K is a measurable function on $X \times X$ such that $\int_X |K(x,y)| d\mu(y) \le C$ for all $x \in X$ and $\int_X |K(x,y)| d\mu(x) \le C$ for all $y \in X$. Define the function Tf by

$$Tf(x) \equiv \int_X K(x,y) f(y) d\mu(y)\,.$$

(a) Show that Tf is well defined almost everywhere and is in $L^p(X)$, and $\|Tf\|_{L^p(X)} \le C\|f\|_{L^p(X)}$. (Hint: Hölder inequality)

(b) Use (a) to show that if $f \in L^1(\mathbf{R}^n)$ and $g \in L^p(\mathbf{R}^n), 1 \le p \le \infty$, then the convolution $f * g \in L^p(\mathbf{R}^n)$ and $\|f * g\|_{L^p(\mathbf{R}^n)} \le \|f\|_{L^1(\mathbf{R}^n)} \|g\|_{L^p(\mathbf{R}^n)}$

(c) Let $\rho_\varepsilon(x) \equiv \varepsilon^{-n} \rho(x/\varepsilon)$, ρ being the same as problem II, show that if $f \in L^\gamma(\mathbf{R}^n)$ $(1 \le \gamma < \infty)$ then $f * \rho_\varepsilon \to f$ in $L^\gamma(\mathbf{R}^n)$ strongly.