PhD Qualify Exam, Analysis, Mar. 3, 2000

Show all works

1. **(a)**[5%] Please state the Hahn-Banach Theorem.

(b)[5%] Please state the Hahn Decomposition Theorem.

2.[10%]Find a representation for the bounded linear functionals on l^p.

3.[10%] Let $< f_n >$ be a sequence of functions in L^p, $1 < p < \infty$, which converge almost everywhere to a function f in L^p, and suppose that there is a constant M such that $\|f_n\| \leq M$ for all n. Then for each function g in L^q we have

$$\int fg = \lim \int f_n g.$$

Is this result true for $p = 1$?

4.[10%] Let g be a nonnegative measurable function on $[0, 1]$. Then

$$\log \int g(t)\, dt \geq \int \log(g(t))\, dt$$

whenever the right side is defined.

5. Suppose that $f : (0, \infty) \to R$ is measurable.

(a)[5%] Prove that if f is integrable and $\epsilon > 0$, then there is a measurable set A in $(0, \infty)$ such that $m(A) < \epsilon$ and $\lim_{x \to \infty} f^{(A)}(x) = 0$, where $f^{(A)}(x) = 0$ if $x \in A$; $f(x)$ if $x \notin A$.

(b)[10%] Prove that if f is integrable on $[0, \infty)$, then for every $\alpha > 1$ there is a set A_α in $(0, \infty)$ such that $m(A_\alpha) = 0$ and $\sum_{n=1}^{\infty} f(n^\alpha x)$ converges for all $x \notin A_\alpha$. (Note: this result is unrelated to part (a).)

(c)[5%] Does the converse of part (b) hold? Prove it or give a counterexample.

6. For $n = 0, 1, 2, ...$, let $f_n(x) = x^n$.

(a)[10%] Let L and L' be bounded linear functionals on $C([0,1])$ such that $L(f_n) = L'(f_n)$ $(n = 0, 1, 2, ...)$. Show that $L = L'$.

(b)[10%] Let L be a bounded linear functional on $C([0,1])$ such that $L(f_n) = \dfrac{1}{n+1}$ $(n = 0, 1, 2, ...)$. Let $f(x) = x^{\frac{1}{3}}$. What is $L(f)$? (Justify your answer.)

7.[10%] Prove or disprove that the 2π periodic functions whose squares have a well defined improper Riemann integral do not define a complete space.

8.[10%] Find the Hausdorff dimension of $C \times C$, where C is the Cantor Set, by computing the following quantities:

First

$$\lambda_\alpha^\epsilon(C \times C) = \inf \sum_{i=1}^{\infty} r_i^\alpha,$$

where $< r_i >$ are radii of sequence of balls $< B_i >$ that covers $C \times C$ and for which $r_i < \epsilon$.

Second,

$$m_\alpha(C \times C) = \lim_{\epsilon \to 0} \lambda_\alpha^\epsilon(C \times C).$$

Finally, Hausdorff dimension of $C \times C$ is $\inf\{\alpha : m_\alpha(C \times C) = \infty\}$.