PhD Qualify Exam, Analysis, Oct. 1, 1999

Show all works

1. **(a)**[5%] Please state the Radon-Nikodym Theorem.

(b)[5%] Please state the Fubini Theorem.

(c)[5%] Please define a Vitali Covering..

(d)[5%] Please state the Vitali Covering Theorem.

2.(a)[10%] Let f be a real-valued bounded measurable function on $[a, b]$, take any $r \in R$, and define the function $F : [a, b] \to R$ by $F(x) = r + \int_a^x f(t)\, dt$. Prove that $F' = f$ *a.e.* and specify each place you invoke any form of the Domination Convergence Theorem.

(b)[10%] Let $C \subset [0, 1]$ be the Cantor ternary set with associated Cantor ternary function f_C and associated Canter ternary measure μ_C (defined by : $\mu_C(a, b] = f_C(b) - f_C(a)$). Prove that μ_C is a continuous measure, *i.e.*, $\mu_C\{x\} = 0$ for each x, and then show that $U \cap C$ is uncountable for every open set U for which $U \cap C \neq \phi$.

3. **(a)**[10%] Given $f \in L^1[0, 1]$. Prove that for all $\epsilon > 0$ there is a $\delta > 0$ such that for every $A \in \mathcal{M}$, for which $m(A) < \delta$, we can conclude that $\int_A |f(t)|\, dm(t) < \epsilon$.

(b) Let X be a compact Hausdorff space and let $C(X)$ be the real Banach space of all real-valued continuous functions on X with *sup*-norm. Prove the following:

(i) [5%] If $L : C(X) \to R$ is a positive linear functional, then L is continuous.

(ii)[5%] If $L : C(X) \to R$ is a continuous linear functional and $\{f_n\} \subset C(X)$ is a *sup*-norm bounded sequence which tends pointwise to a function $f \in C(X)$, then $\lim L(f_n) = L(f)$.

4. **(a)**[10%] Assume the inequality

$$\|fg\|_1 \leq \|f\|_p \|g\|_q$$

holds for all functions $f \in L^p$ and $g \in L^q$. Show that the relation between p and q is $\frac{1}{p} + \frac{1}{q} = 1$. (Hint: Consider $f_\lambda(x) = f(\lambda x)$ and $g_\lambda(x) = g(\lambda x)$.)

(b)[10%] Assume that $f_n \to f$ in measure. (Definition of convergence in measure: for each $\epsilon > 0$, there is an $N > 0$ such that $\mu\{x : |f_n(x) - f(x)| > \epsilon\} < \epsilon$, for all $n > N$.) Assume that $|f_n| \leq g$, $g \in L^p(\mu)$, $f \in L^p(\mu)$, and μ is a σ-finite measure on X. Prove that $f_n \to f \quad \text{in} \quad L^P(\mu)$.

5. Suppose that $x_1, x_2, x_3, \cdots$ is a sequence of points in the unit interval $[0,1]$ such that for every continuous real valued function f defined on $[0,1]$, $\lim \frac{1}{n}[f(x_1) + \cdots + f(x_n)]$ exists. Define this limit to be $L(f)$.

(a)[10%] Prove that there is a positive measure μ, defined on the σ-algebra of all Borel sets of $[0,1]$, such that

$$L(f) = \int_{[0,1]} f \, d\mu$$

for all continuous functions f. State any theorems you use.

(b)[10%] Prove that the measure μ in part (a) is Lebesgue measure if and only if for every integer $k \geq 1$, $L(x^k) = \frac{1}{k+1}$, *i.e.*,

$$\frac{1}{n}(x_1^k + \cdots + x_n^k) \to \frac{1}{k+1} \quad \text{as} \quad n \to \infty.$$

State any theorems you use.