PhD Qualify Exam, Analysis, Feb. 26, 1999

Answer any Five questions and only five. Show all works

1. Let $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Let $f \in L^p([0,1])$ and let $\{f_n\}$ be a sequence in $L^p([0,1])$ so that $\|f_n\|_p \leq 1$ for all n and $\lim_{n\to\infty} \int_E f_n = \int_E f$ for every measurable subset E of $[0,1]$.

(a)[10%] Prove that $\lim_{n\to\infty} \int_0^1 f_n g = \int_0^1 fg$ for every $g \in L^q([0,1])$.

(b)[10%] Does $\lim_{n\to\infty} f_n(x) = f(x)$ a.e. $x \in [0,1]$? Prove it or give a counterexample.

2.[20%] Let X be a metric space. A sequence $\{f_n\}$ of real-value functions on X is said to converge <u>continuously</u> to the function $f : X \longrightarrow R$ at the point $x \in X$ if for every sequence $\{x_n\} \subseteq X$ which converges to x one has $\lim_{n\to\infty} f_n(x_n) = f(x)$.

(a)[10%] Prove that if the sequence $\{f_n\}$ converges continuously to f at every point of X, then f is continuous on X.

(b)[10%] Prove that if X is compact and if $\{f_n\}$ congerges continuously to f at every point of X, then $\{f_n\}$ converges uniformly on X to f.

3.[20%] Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of nonnegative measurable functions on R such that $\lim_{n\to\infty} f_n(x) = f(x)$ a.e. in x. For each part below, determine whether the additional assumptions made are enough to conclude that $\lim_{n\to\infty} \int_R f_n = \int_R f$ (including the possibility $\infty = \infty$).

Justify your answer by explaining why it can be applied or by giving a counterexample.

(a) $f_n(x) \leq 1$ for all n and x, and $\{x : \ f_n(x) \neq 0\}$ has finite measure for every n.

(b) $f_n(x) \leq f(x)$ for all n and x.

(c) $\lim_{n\to\infty} \int_E f_n = \int_E f$ for every E of finite measure.

4. **(a)**[10%] Assume that $\{f_n\} \longrightarrow f$ uniformly and $\{g_n\} \longrightarrow g$ uniformly. Does $\{f_n g_n\} \longrightarrow fg$ uniformly? Prove it or give a counterexample.

(b)[10%] Let $\{f_n\}$ be a sequence of absolutely continuous functions and $\{f_n\} \longrightarrow f$ uniformly. Is f absolutely continuous?

5. **(a)**[10%] State the Riesz Representation Theorem (for bounded linear functionals on L^p).

(b)[10%] Assume the following inequality holds: $\|f\|_{L^4} \leq C\|S\widehat{f}\|_{L^2}$,

where C is a constant, S is some function of x, and $\widehat{f}$ denote the Fourier transform. We also take the *Parseval formula* for granted. $\int_{-\infty}^{\infty} f(x)\overline{g(x)}dx = \int_{-\infty}^{\infty} \widehat{f}(t)\overline{\widehat{g}(t)}dt$

holds for all $f \in L^2$ and $g \in L^2$. Use Riesz Representation Theorem and Parseval Formula to prove the following inequality. $\|\frac{\widehat{g}}{S}\|_{L^2} \leq C\|g\|_{L^{\frac{4}{3}}}$.

6. **(a)**[4%] State the definitions of the following notions: first category and nowhere dense.

(b)[4%] State the Baire category theorem.

(c)[12%] Prove or disprove the following statements.

(i) All sets of first category are nowhere dense.

(ii) All sets of first category in [0,1] have Lebesgue measure less than 1.

7.[20%] Define that f_n converges weakly to f, where $f, f_n \in L^1[0,1]$, if $\forall g \in L^\infty[0,1]$,

$$\lim_{n\to\infty} \int_0^1 (f_n(x) - f(x))g(x)dx = 0.$$

Given $f, f_n \in L^1[0,1]$. Prove that f_n converges weakly to f if and only if

$$\sup_n \|f_n\|_1 < \infty \qquad \text{and} \qquad \lim_{n\to\infty} \int_E (f_n(x) - f(x))dx = 0$$

for every measurable set E.