Qualifying Exam in Partial Differential Equations   March, 2024

E: Easy M: Moderate D: Difficult

1. (E, 10%; 2021, Fall) Let $f \in C^1(\mathbb{R}^n)$ and suppose that for each open ball B, there exists a solution of the bounded value problem
$$\begin{cases} -\Delta u = f & \text{in } B \\ \frac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } \partial B, \end{cases}$$
where $\mathbf{n}$ is the outward unit normal vector field to ∂B. Show that $f \equiv 0$.

2. (E, 10%; 2018, Spring) Solve the first order linear equation
$$yu_x + u_y = u, \quad (x, y) \in \mathbb{R}^2,$$
with $u(0, x) = x^2$.

3. (E, 10%) Let $U \subset \mathbb{R}$ be an interval and assume u, $v \in L^1_{loc}(U)$. We say that v is the weak derivative of u if
$$\int_U u(x)\varphi'(x)\,dx = -\int_U v(x)\varphi(x)\,dx \quad \text{for all } \varphi \in C_0^\infty(U).$$
Let
$$u(x) = \begin{cases} x, & 0 < x \le 1, \\ 2, & 1 < x < 2. \end{cases}$$
Does u have a weak derivative over the domain $U = (0, 2)$?

4. (E, 15%; 2022, Fall) Solve the initial value problem
$$\begin{aligned} u_{tt} &= u_{xx} + \cos x, && -\infty < x < \infty, \quad t > 0, \\ u(x, 0) &= \sin x, && -\infty < x < \infty, \\ u_t(x, 0) &= 1 + x, && -\infty < x < \infty. \end{aligned}$$

5. (E, 20%; 2023, Spring) Let $f \in C^1([0, L])$ be a real-valued function with $f(0) = f(L) = 0$. Show that
$$\int_0^L f^2 dx \le \left(\frac{L}{\pi}\right)^2 \int_0^L \left(\frac{df}{dx}\right)^2 dx,$$
and the equality holds if and only if
$$f(x) = c\sin\left(\frac{\pi x}{L}\right),$$
for all $x \in [0, L]$, where c is a constant.

6. (M, 20%; 2022, Fall) Use the Fourier transform method to solve the initial value problem
$$\begin{cases} u_t = u_{xx}, & -\infty < x < \infty, \ t > 0, \\ u(x, 0) = f(x), & -\infty < x < \infty, \end{cases}$$
where $f \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$, and show that $u(x, t)$ satisfies the following inequality
$$\|u\|_p(t) \le \frac{1}{(4\pi t)^{\frac{1}{2}\left(\frac{1}{q} - \frac{1}{p}\right)}} \|f\|_q, \quad t > 0,$$
for $1 \le q \le p \le \infty$. (Note that the L^p, L^q norms are with respect to the variable x.)

7. (M, 15%; 2018, Spring) Find a traveling wave equation of the viscous Burgers' equation

$$u_t + uu_x = u_{xx}, \quad x \in \mathbb{R}, \quad t > 0.$$

That is, $u(x,t) = v(x - ct)$ with $c \in \mathbb{R}$ and $v \in C^2(\mathbb{R})$ such that $v(s) \to 0$ as $s \to -\infty$ and $v(s) \to 1$ as $s \to \infty$.