Qualifying Examination in General Algebra

March 2024

- Attempt all problems. Show all your work and justify all your answers.
- Easier: 1, 2, 3, 6; Medium: 4, 5; Harder: 7

1. (15 points) Let G be a simple group of order 168. Determine the number of elements of order 7 in G.

2. (15 points) Let G be a nonabelian group of order p^3, where p is a prime. Prove that the center of G is of order p.

3. (10 points) Give an example of a nonzero homomorphism $f\colon R \to S$ of rings with identity such that $f(1_R) \neq 1_S$. Here 1_R and 1_S denote the multiplicative identities of R and S, respectively.

4. (15 points) Let R be a commutative ring with identity $1_R \neq 0$, and let $R[x]$ be the ring of polynomials in x over R. Let $f = \sum_{i=0}^{n} a_i x^i \in R[x]$, where n is a positive integer. Prove that if f is a unit in $R[x]$, then a_0 is a unit in R and $a_1, \ldots, a_n$ are nilpotent elements of R.

5. (15 points) Let n be a positive integer, and let R be a commutative ring with identity $1_R \neq 0$. Suppose F is a free R-module of rank n with basis $\{v_1, \ldots, v_n\}$ and A is a nonzero R-module. Prove that every element of $A \otimes_R F$ can be written uniquely in the form $\sum_{i=1}^{n} a_i \otimes v_i$, where $a_1, \ldots, a_n \in A$.

6. (15 points) Let E, F, and K be fields. Prove that if E is an algebraic extension of F and F is an algebraic extension of K, then E is an algebraic extension of K.

7. (15 points) Let $\mathbb{C}$ denote the field of complex numbers. Suppose t is transcendental over $\mathbb{C}$. Is $\mathbb{C}(t, \sqrt{1-t^2})$ a purely transcendental extension of $\mathbb{C}$?