Qualifying Examination: Partial Differentiation Equations

Autumn, 2023

Do all the problems and show all your work. (E: easy, M: moderate, D:difficult)

1. (E, 10 points)
   Let $g(t)$ be a C^1 function defined in the interval $[a, b]$. Please prove that

$$\lim_{n\to\infty} \int_a^b g(t)\sin(nt)\ dt = 0.$$

2. (E, 15 points)
   Please solve the Cauchy problem

$$\begin{cases} uu_x + yu_y = x, & (x, y) \in R^2, \\ u(x, 1) = 2x. \end{cases}$$

3. (M, 15 points)
   Please use the Fourier transform method to solve the initial value problem

$$u_t = Ku_{xx}, \quad -\infty < x < \infty, \quad t > 0,$$

and

$$u(x, 0) = \begin{cases} a_1, & x < 0, \\ a_2, & x > 0. \end{cases}$$

Here a_1, a_2 and K are given constants.
Is the solution unique? And please find $\lim_{t\to\infty} u(x, t)$.

4. (M, 15 points)
   Consider the wave equation

$$\begin{cases} u_{tt} = c^2(u_{xx} + u_{yy} + u_{zz}) \\ u(X, 0) = \phi(X), \\ u_t(X, 0) = \psi(X). \end{cases}$$

Here $X = (x, y, z)$ is in R^3. Kirchhoff's formula gives the solution

$$u(X_0, t_0) = \frac{1}{4\pi c^2 t_0} \int\int_S \psi(X) dS + \frac{\partial}{\partial t_0}\left[\frac{1}{4\pi c^2 t_0} \int\int_S \phi(X) dS\right],$$

where S is the sphere of center X_0 with radius ct_0.

(a) What is Huygens' Principle?

(b) Please explain why Huygens' Principle is not valid in two dimensional spaces.

(c) If ϕ and ψ vanish outside a sphere with radius R, where does u has to vanish?

5. (D, 15 points)

Let $u(x)$ be a real-valued function defined in R^3. $x = (x_1, x_2, x_3)$ and $Du(x) = (\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \frac{\partial u}{\partial x_3})$.

Assume that $p \in [1, 3)$. p^* is defined as $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{3}$.

Please prove that there exists a constant C, depending only on p, such that

$$\|u\|_{L^{p^*}(R^3)} \leq C\|Du\|_{L^p(R^3)}$$

for all $u \in C_c^1(R^3)$.

6. (E, 15 points)

(a) Let $u(x, y)$ be a harmonic function defined in a set D of R^2. Please state the maximum principle for u. (You don't need to prove it.)

(b) A function $u(x, y)$ is subharmonic in D if $\Delta u \geq 0$. Is the maximum principle true for subharmonic functions?

(c) Is the maximum principle true for the wave euqations?

7. (M, 15 points) $f(x)$ and $g(x)$ are continuous functions satisfying $f(x + 1) = f(x)$ and $g(x + 1) = g(x)$. Please prove that

$$\lim_{n\to\infty} \int_0^1 f(x)g(nx)dx = \left(\int_0^1 f(x)dx\right)\left(\int_0^1 g(x)dx\right)$$