**Qualifying Exam – Differential Geometry**
**Fall 2023**

1. Let $M$ and $N$ be smooth manifolds, and let $f : M \to N$ be a smooth map.

   (a) (10 points) (Easy) Show that if $f$ is a submersion, then $f$ is an open map.

   (b) (10 points) (Easy) Show that if $M$ and $N$ have the same dimension and $f$ is an immersion, then $f$ is a local diffeomorphism.

2. Let $M$ be a smooth manifold. A critical point of a smooth function $f : M \to \mathbb{R}$ is a point $p \in M$ such that $df_p = 0$. Let $T_pM$ be the tangent space to $M$ at $p$.

   (a) (10 points) (Easy) Let $p$ be a critical point of $f$. Define $H : T_pM \times T_pM \to \mathbb{R}$ by

   $$H(v, w) = XYf(p),$$

   where $X$, $Y$ are smooth vector fields on $M$ and $X_p = v$, $Y_p = w$. Show that $H$ is well-defined, bilinear, and symmetric.

   (b) (10 points) (Medium) Let $\gamma : \mathbb{R} \to M$ be a curve such that $\gamma(0) = p$ and $\gamma'(0) = v$. Show that

   $$H(v, v) = \frac{d^2(f \circ \gamma)}{dt^2}(0).$$

3. (15 points) (Medium) Given the fact that every vector field on $S^2$ must vanish somewhere, show that $S^2$ has no Lie group structure.

4. (15 points) (Medium) Consider the map $\hat{F} : S^2 \subset \mathbb{R}^3 \to \mathbb{R}^6$ given by

   $$\hat{F} : (x, y, z) \mapsto (x^2, y^2, z^2, \sqrt{2}yz, \sqrt{2}zx, \sqrt{2}xy).$$

   Prove that $\hat{F}$ gives rise to a smooth embedding $F : \mathbb{R}P^2 \to \mathbb{R}^6$.

5. Consider a smooth map $f : S^3 \to S^2$.

   (a) (10 points) (Easy) Let $\alpha$ be a 2-form on $S^2$ such that $\int_{S^2} \alpha = 1$. Show that there exists a 1-form $\eta$ on $S^3$ such that $f^*\alpha = d\eta$.

   (b) (10 points) (Easy) Show that the value of the integral $\int_{S^3} \eta \wedge d\eta$ is independent of the choices of $\alpha$ and $\eta$. (Hence it depends only on $f$, and is called the **Hopf invariant** of $f$.)

   (c) (10 points) (Medium) Show that the Hopf invariant of $f$ depends only on the homotopy class of $f$.