Ph.D Qualify Exam **Analysis** **2023 Fall**

E: Easy; M: Moderate; D: Difficult

1(E, 10%, 2019, Spring). A sequence $\{f_n\}$ of Lebesgue measurable functions is called Cauchy sequence in measure if given $\varepsilon > 0$ there is N such that

$$\text{Leb}\left(\{x \mid |f_n(x) - f_m(x)| \geq \varepsilon\}\right) < \varepsilon$$

for all $m, n > N$, where Leb$(\cdot)$ represents the Lebesgue measure. (a) Write down the definition of the convergence in measure. (b) Prove that $\{f_n\}$ converges in measure.

2(E, 15%, 2020, Fall). Let $k(x, y)$ be a measurable function on $\mathbb{R}^n \times \mathbb{R}^n$ satisfying that

$$\int_{\mathbb{R}^n} |k(x,y)| dy \leq C \text{ for a.e. } x \text{ and } \int_{\mathbb{R}^n} |k(x,y)| dx \leq C \text{ for a.e. } y,$$

where $C > 0$ is a universal constant. Prove that

$$(Tf)(x) := \int_{\mathbb{R}^n} k(x,y) f(y) dy$$

is a bounded operator on $L^p(\mathbb{R}^n)$ with $\|Tf\|_p \leq C\|f\|_p$ for $1 \leq p \leq \infty$.

3(E, 15%, 2021, Spring). Let $f \in L^1([0,\infty))$ and $a > 0$. Show that

$$\int_0^\infty \int_0^\infty \sin(ax) f(y) e^{-xy}\, dy\, dx = a \int_0^\infty \frac{f(y)}{a^2 + y^2}\, dy.$$

4(E, 10%, 2021, Fall). Find the value of the integral

$$\int_0^\infty \frac{\sin x}{x} e^{-x} dx.$$

5(E, 10%). Let f and $\{f_n\}_{n\in\mathbb{N}}$ be Lebesgue integrable functions defined on $(-\infty, \infty)$. Suppose that

$$\int_{-\infty}^\infty |f_n(x) - f(x)|^2\, dx \leq \frac{1}{n^{3/2}} \text{ for } n \geq 1.$$

Does $f_n(x)$ converge to $f(x)$ almost everywhere when $n \to \infty$? If your answer is YES, please prove it.

6(E, 10%). Let $h : [0,\infty) \to \mathbb{R}$ be a Lebesgue integrable function. Find the value of

$$\lim_{n\to\infty} \int_0^\infty x^n e^{-2nx} h(x) dx$$

and prove it.

7(M, 10%). Let $a > 0$ and let m be a measure defined by

$$m(E) = \int_E \frac{1}{x^2 + a^2} dx$$

for any Lebesgue measurable set E. Find the Radon-Nikodym derivative dx/dm.

8(M, 10%). Suppose that $\{f_n\}_{n\in\mathbb{N}}$ is a uniformly bounded sequence and $f_n \to f$ almost everywhere. On the other hand, let $\{g_n\}_{n\in\mathbb{N}}$ be a sequence in $L^2(\mathbb{R})$ satisfying $g_n \to g$ in $L^2(\mathbb{R})$ as $n \to \infty$. Prove that $\int_{\mathbb{R}} |f_n(x)g_n(x) - f(x)g(x)|^2 dx \to 0$ as $n \to \infty$.

9(M, 10%). (a) Prove that

$$\left(1 + \frac{x}{n}\right)^n \leq e^x$$

for any $n \in \mathbb{N}$ and $x \geq 0$. (b) Find the value of

$$\lim_{n\to\infty} \int_0^n \left(1 + \frac{x}{n}\right)^n e^{-2x} dx$$

and prove it.