Qualifying Examination in General Algebra

September 2023

- Attempt all problems. Show all your work and justify all your answers.
- Easier: 1, 2, 3; Medium: 4, 5, 6; Harder: 7
- $\mathbb{Z}$ denotes the ring of integers, and $\mathbb{Q}$ denotes the field of rational numbers.

1. (10 points) Let G be a group, and let G' be the commutator subgroup of G. Suppose H is a normal subgroup of G. Prove that G/H is abelian if and only if $G' \subseteq H$.
2. (15 points) Let p be an odd prime. Classify up to isomorphism all groups of order $2p$.
3. (15 points) Let E, F, and K be fields. Prove that if E is an algebraic extension of F and F is an algebraic extension of K, then E is an algebraic extension of K.
4. (15 points) Let $\mathbb{Q}[x, y]$ be the ring of polynomials in the variables x and y with coefficients in $\mathbb{Q}$. Determine if the rings $\mathbb{Q}[x, y]/(x^2 - y)$ and $\mathbb{Q}[x, y]/(x^2 - y^4)$ are isomorphic.
5. (15 points) Find the Galois group of the polynomial $x^4 - 3$ over $\mathbb{Q}$.
6. (15 points) Let R be a commutative ring with identity $1_R \neq 0$, and let A and B be R-modules. We denote by $\mathrm{Hom}_R(A, B)$ the R-module consisting of all R-module homomorphisms from A to B. Let V be a free R-module of finite rank and $V^* = \mathrm{Hom}_R(V, R)$. Prove that there is a canonical isomorphism

$$\mathrm{Hom}_R(A \otimes_R V, B) \cong \mathrm{Hom}_R(A, V^* \otimes_R B)$$

   of R-modules.
7. (15 points) Let $\mathbb{Z}[x]$ be the ring of polynomials in the variable x with coefficients in $\mathbb{Z}$. Determine all prime ideals $\mathfrak{p}$ of $\mathbb{Z}[x]$ such that $\mathfrak{p} \cap \mathbb{Z} \neq \{0\}$.