Qualifying Exam in Differential Equations

March 2023

Solve all problems

(1) (E, Old problem, 20 points) Let $f : [-r, r] \to \mathbb{R}$ be a C^1 function with

$$A = \frac{1}{2r} \int_{-r}^{r} f(y) dy \,.$$

Prove that

$$\int_{-r}^{r} (f(x) - A)^2 \, dx \leq (2r)^2 \int_{-r}^{r} (f'(x))^2 \, dx \,.$$

(2) (E, Old problem, 20 points) Let $x \in \mathbb{R}^n$, prove that the Laplace equation $\Delta u(x) = 0$ is rotation invariant.

(3) Let $a \in \mathbb{R}$ and $f(t, x)$ satisfy the equation $\partial_t f + a\partial_x f = \partial_x^2 f$ with initial condition $f(0, x) = u_0(x)$, $-\infty < x < \infty$.
a. (E, 10 points) Use the Fourier transform method to solve f.
b. (D, 15 points) If $b \in \mathbb{R}$ and g satisfy the equation $\partial_t g + b\partial_x g = \partial_x^2 g$ with initial condition $g(0, x) = u_0(x)$. Assume that $u_0 \in L^1_x$, prove that

$$\|f - g\|_{L^\infty_x} \leq C|b - a| \|u_0\|_{L^1_x}$$

for some constant $C > 0$ independent of u_0 and a, b.

(4) (M, 15 points) Let $t > 0, x \in \mathbb{R}^3, v \in \mathbb{R}^3$ and $f(t, x, v)$ satisfy the transport equation $\partial_t f + v \cdot \nabla_x f = 0$ with initial condition $f(0, x, v) = f_0(x, v)$. Prove that

$$\left\| \|f\|_{L^1_v} \right\|_{L^\infty_x} \leq Ct^{-3} \left\| \|f_0\|_{L^\infty_v} \right\|_{L^1_x}$$

for some constant $C > 0$.

(5) (M, 20 points) Let u solves the one dimensional wave equation

$$\begin{cases} u_{tt} - u_{xx} = 0 \,, & x \in \mathbb{R}, t > 0 \\ u(0, x) = g(x) \,, u_t(0, x) = h(x) \,, & x \in \mathbb{R} \,, \end{cases}$$

where g and h are smooth and have compact support. Show that

$$\int_{-\infty}^{\infty} u_t^2(x, t) dx = \int_{-\infty}^{\infty} u_x^2(x, t) dx$$

for large enough time t.