PhD qualifying exam on Mathematical Programming   March the 2nd, 2023

1. (30 points, Revised from March the 1st 2013) Consider the bilinear program to minimize $f(x, y) = c^T x + d^T y + x^T Hy$ subject to $x \in X \subset \mathbb{R}^n,\ y \in Y \subset \mathbb{R}^m$, where X and Y are bounded polyhedral sets and H is a $n \times m$ matrix, $c \in \mathbb{R}^n;\ d \in \mathbb{R}^m$. Let $\hat{x}$ and $\hat{y}$ represent extreme points (vertices) of the polyhedral sets X and Y, respectively.

   (a) (3 points, old) Verify that the objective function is neither quasiconvex nor quasiconcave.

   (b) (5 points, new) Show that, the bilinear program is equivalent to a piecewise linear concave minimization problem with linear constraints.

   (c) (5 points, old. See also from September 28th, 2012) Show that, the optimum for minimizing a concave function over a bounded polyhedral set P must be achieved at least on one of the extreme points of P.

   (d) (5 points, old) Prove that there exists an extreme point $(\bar{x}, \bar{y})$ that solves the bilinear program.

   (e) (7 points, new) When can a quadratic minimization problem $\phi(x) = x^T Qx + 2q^T x + q_0$ be reduced to a bilinear problem? Verify your answer and show the reduction. (Q is a $n \times n$ symmetric matrix; $q \in \mathbb{R}^n,\ q_0 \in \mathbb{R}$.)

   (f) (5 points, old) Prove that the point $(\hat{x}, \hat{y})$ is a local minimum of the bilinear program if and only if the following are true:

      (i) $(\forall x \in X,\ \forall y \in Y)\ c^T(x - \hat{x}) \geq 0$ and $d^T(y - \hat{y}) \geq 0$.

      (ii) $c^T(x - \hat{x}) + d^T(y - \hat{y}) > 0$, whenever $(x - \hat{x})^T H(y - \hat{y}) < 0$.

2. (15 points, new, comprehensive) Show that, any concave minimization problem with a piecewise linear separable objective function can be reduced to a bilinear program. You may set up the objective function as $\Psi(x) = \sum_{i=1}^{n} \psi_i(x_i),\ x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$ where $\psi_i(x_i)$ is a concave piecewise linear function of only one component x_i. You also need to show that, an optimal solution to the reduced equivalent bilinear program (of your own design) provides an optimal solution to minimize $\Psi(x)$.

3. (15 points, old, from September 28th, 2012) Recall that linear programming (LP) is a special case of the following *conic optimization* model

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \in \mathcal{K}, \end{array}$$

   where $\mathcal{K} \subseteq E^n$ is a prescribed closed convex cone. For example, $\mathcal{K} = \{x | x \geq 0\}$. Here we assume that A, b, c are of proper dimensions and the rows in A are linearly independent.

   When $\mathcal{K} \triangleq \mathcal{K}_L$, the conic optimization model becomes the so-called "Second Order Cone Programming (SOCP)." When $\mathcal{K} \triangleq \text{PSD}(n)$, the conic optimization model becomes the so-called "Semidefinite Programming (SDP)." The popularity of SOCP is also due to that it is a generalized form of convex QCQP (Quadratically Constrained Quadratic Programming). To be precise, consider the following convex QCQP:

$$\begin{array}{ll} \text{minimize} & x^T Q_0 x + 2b_0^T x \\ \text{subject to} & x^T Q_i x + 2b_i^T x + c_i \le 0, \ i = 1, \ldots, m, \end{array}$$

where $Q_i \succeq 0$, i.e., Q_i is positive semidefinite for $i = 0, 1, \cdots, m$.

(a) (5 points) Given that $t \in E^1$ and $x \in E^n$, prove that

$$t \ge x^T x \ \text{ if and only if } \ \left\| \begin{pmatrix} \frac{t-1}{2} \\ x \end{pmatrix} \right\| \le \tfrac{t+1}{2}.$$

(b) (10 points) Using the result of (a), please formulate a convex QCQP as an SOCP problem (P).

4. (10 points, new, comprehensive) Compute the optimal value of the following convex QCQP, where $x_1, \ x_2 \in \mathbb{R}^n$.

$$\begin{array}{ll} \min & f_0(x) = x_1^2 + x_2 \\ \text{s.t.} & f_1(x) = x_1^2 - x_1 x_2 + 2x_2^2 - x_3 \le 0, \\ & f_2(x) = x_1^2 - x_2 - 1 \le 0. \end{array}$$

5. (7 points, old, from September 28th, 2007) Let

$$f(x) = \frac{1}{p}\|x\|^p, \ p > 1, \ x \in \mathbb{R}^n.$$

Compute the conjugate function f^* and verify that $f^{**} = f$.

6. (23 points, new, standard) Given that n is a positive integer, let us denote the n-dimensional Euclidean space by E^n. A subset $\mathcal{K}$ of E^n is called a *pointed convex cone* in E^n if the following conditions are satisfied:

i. The origin of the space, i.e., vector 0, belongs to $\mathcal{K}$.

ii. If $x \in \mathcal{K}$ and $-x \in \mathcal{K}$ then $x = 0$.

iii. If $x \in \mathcal{K}$ then $tx \in \mathcal{K}$ for all $t > 0$.

iv. If $x \in \mathcal{K}$ and $y \in \mathcal{K}$ then $x + y \in \mathcal{K}$.

Let us denote the standard Lorentz cone as follows:

$$\mathrm{SOC}(n+1) = \left\{ \begin{pmatrix} t \\ x \end{pmatrix} \middle| \ t \in E^1, \ x \in E^n, \ t \ge \|x\| \right\}.$$

Given that $d_1, d_2, \ldots, d_p$ are known positive integers, we let $\mathcal{K}_L \subseteq E^{\left(\sum_{i=1}^{p} d_i + p\right)}$ be the Cartesian product of Lorentz cones, i.e.,

$$\mathcal{K}_L = \mathrm{SOC}(d_1 + 1) \times \cdots \times \mathrm{SOC}(d_p + 1).$$

In addition, we denote the set of all n-by-n real symmetric matrices by $S^{n \times n}$ and let $\mathrm{PSD}(n) \subseteq S^{n \times n}$ be the set of positive semidefinite matrices in $S^{n \times n}$. In the following, if C is a subset in E^p and C^* is its polar cone, then the dual cone C^0 of C is the negative of C^*. That is, $C^0 = -C^*$.

(a) (10 points) Which ones, $\mathrm{SOC}(n + 1)$, $\mathcal{K}_L$, and $\mathrm{PSD}(n)$ are closed pointed convex cones? Verify your answers.

(b) (13 points) Find the dual cones of the above three sets in their respective domain.