PhD Qualify Exam
General Analysis

March 2023

E: Easy, M: Moderate, D: Difficult

1. (E, 10 points) (Sept. 2011) Let f be Lebesgue measurable on $[0, 1]$. Assume that

$$\int_0^1 [f(x)]^m dx = c, \text{ for all } m \in \mathbb{N}$$

where c is some constant. Show that $f = \chi_A$ a.e. for some $A \subset [0, 1]$.

2. (E, 20 points) (March 2010) Let $f \in L^1(\mathbb{R})$. Define

$$F(x) = \int_{\mathbb{R}} f(t) \frac{\sin xt}{t} \, dt.$$

   (a) Prove that F is differentiable on $\mathbb{R}$ and find $F'(x)$

   (b) Determine whether or not F is absolutely continuous on every compact interval of $\mathbb{R}$.

3. (M, 15 points) (October 2015) Let $1 \leq p < \infty$ and g be an integral function on $[0, 1]$, suppose that there exists $M > 0$ such that

$$\left| \int_0^1 fg \, dx \right| \leq M \|f\|_p$$

for all bounded measurable function f, then $g \in L^q$ and $\|g\|_q \leq M$, weher $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

4. (E, 15 points) (March 2014) Let E be a measurable set in $\mathbb{R}^n$. f and f_k are measurable in E. If $p > 0$, and $\int_E |f - f_k|^p \to 0$ as $k \to \infty$, show that there exists a subsequence $f_{k_j} \to f$ a.e. in E.

5. (E, 10 points) Let $\{f_n\}$ be a sequence of Lebesgue measurable functions on the interval $(0, 1)$ so that,

   (i) $\sup_n \int_0^1 |f_n| \, dx < \infty$

   (ii) $f_n \to 0$ in measure

Show that

$$\int_0^1 \sqrt{|f_n|} \, dx \to 0 \quad \text{as } n \to \infty.$$

6. (M. 15 points) Let $f, g : (0,1) \to [0,\infty)$ be measurable functions. For each Lebesgue measurable set $A \subset (0,1)$, let $m(A)$ be the Lebesgue measure of A. Prove that if

$$m\Big(\big\{x \in (0,1) \mid g(x) > \alpha\big\}\Big) \leq \int_{\{x\in(0,1) \mid f(x)>\alpha\}} f(x)\, dx$$

for every $\alpha > 0$, then

$$\int_0^1 \big[g(x)\big]^p\, dx \leq \int_0^1 \big[f(x)\big]^{p+1}\, dx$$

for every $p > 0$.

7. (E, 15 points) Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of real-valued integrable functions on the unit interval $[0,1]$ converging pointwise almost everywhere to a function f on $[0,1]$. Assume that

$$\sup_n \int_0^1 |f_n| \max\big(0, \log|f_n|\big)\, dx < \infty$$

Show that f is integrable and

$$\lim_{n\to\infty} \int_0^1 f_n\, dx = \int_0^1 f\, dx.$$