Qualifying Examination: Partial Differentiation Equations

Oct., 2022

Do all the problems and show all your work. (E: easy, M: moderate, D:difficult)

1. (E, 20 points)
   Consider the transport equation:

$$\frac{\partial u}{\partial t} + x\frac{\partial u}{\partial y} = 0, \quad (x,y) \in R^2, \ t > 0$$
$$u(x,y,0) = u_0(x,y),$$

   where u_0 is a continuous function with $u_0(x,y) = 0$ for $|x| + |y| \geq 1$.

   (a) Please solve the initial value problem.

   (b) Please prove that for fixed $t_0 > 0$ and $x_0 \in R$,

$$\lim_{y\to\infty} u(x_0, y, t_0) = 0.$$

2. (M, 20 points)
   Please use the Fourier transform method to solve the initial value problem

$$\begin{aligned} u_t &= u_{xx}, \quad -\infty < x < \infty, \quad t > 0, \\ u(x,0) &= f(x), \quad -\infty < x < \infty. \end{aligned}$$

   And prove that $u(x,t)$ satisfies the following inequality

$$\|u\|_p(t) \leq \frac{1}{(4\pi t)^{\frac{1}{2}(\frac{1}{q}-\frac{1}{p})}}\|f\|_q, \quad t > 0,$$

   for $1 \leq q \leq p \leq \infty$. (Note that the L^p, L^q norms are with respect to x.)

3. (D, 20 points)
   Let $u(x)$ be a real-valued function defined in R^3. $x = (x_1, x_2, x_3)$ and $Du(x) = (\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \frac{\partial u}{\partial x_3})$.

   Assume that $p \in [1,3)$. p^* is defined as $p^* = \frac{1}{p} - \frac{1}{3}$.

   Please prove that there exists a constant C, depending only on p, such that

$$\|u\|_{L^{p^*}(R^3)} \leq C\|Du\|_{L^p(R^3)}$$

   for all $u \in C_c^1(R^3)$.

4. (M, 20 points)
Let D be a connected bounded open set in R^2. $u(x,y)$ is a harmonic function in D that is continuous on $\bar{D}$.

   (a) Please state the Maximum Principle and prove it.

   (b) Let $u(x,y)$ and $v(x,y)$ be harmonic functions in D that are continuous on $\bar{D}$. And $u(x,y) \geq v(x,y)$ on ∂D. Please show that either $u > v$ in D or $u = v$ on $\bar{D}$.

5. (E, 20 points)
Please solve the initial value problem:

$$\begin{aligned} u_{tt} &= u_{xx} + \cos x, \quad -\infty < x < \infty, \quad t > 0, \\ u(x,0) &= \sin x, \quad -\infty < x < \infty. \\ u_t(x,0) &= 1 + x. \end{aligned}$$