PhD Qualify Exam

General Analysis

(E: Easy, M: Moderate, D: Difficult) October 13, 2022

1. (E, 15%) Let $E \subset \mathbb{R}^n$. If $p > 0$, $\int_E |f_k - f|^p \to 0$, and $\int_E |f_k|^p \leq M$ for $k \in \mathbb{N}$, show that $\int_E |f|^p \leq M$.

2. (E, 15%, 2013, March) Let $\{f_k\}$ be a sequence of measurable functions defined on a measurable subset E of $\mathbb{R}^n$ with $|E| < +\infty$ .If $|f_k(x)| \leq M_x < +\infty$ for all k for each $x \in E$, show that given $\epsilon > 0$, there is a closed $F \subset E$ and a finite M such that $|E \backslash F| < \epsilon$ and $|f_k(x)| \leq M$ for all k and all $x \in F$.

3. (M, 15%, 2018, March) Let $E \subset \mathbb{R}^n$. If $f_k \to f$ in $L^p(E)$, $1 \leq p < \infty$, $g_k \to g$ pointwise in E, and $\|g_k\|_\infty \leq M$ for all $k \in \mathbb{N}$, prove that $f_k g_k \to fg$ in $L^p(E)$.

4. (M, 15%, 2005, September) Let $\{f_n\}$ be a sequence of functions in $L^p[0,1], 1 < p < \infty$, which converge almost everywhere to a function f in $L^p[0,1]$, and suppose that there is a constant M such that $\|f_n\|_p \leq M$ for all n. For each function g in $L^q[0,1]$ and $\frac{1}{p} + \frac{1}{q} = 1$, show that
$$\int_0^1 fg = \lim_{n\to\infty} \int_0^1 f_n g.$$

5. (M, 20%) Suppose that E is a subset of $\mathbb{R}^n$ with $|E|_e < +\infty$. Then E is measurable if and only if given $\epsilon > 0$, $E = (S \cup N_1) \setminus N_2$, where S is a finite union of nonoverlapping intervals and $|N_1|_e, |N_2|_e < \epsilon$.

6. (D, 20%) A sequence $\{\phi_k\}$ of set functions is said to be uniformly absolutely continuous if given $\epsilon > 0$, there exists $\delta > 0$ such that if E satisfies $|E| < \delta$, then $|\phi_k(E)| < \epsilon$ for all k. If $\{f_k\}$ is a sequence of integrable functions on $(0,1)$ which converges pointwise a.e. to an integrable f, show that $\int_0^1 |f - f_k| \to 0$ if and only if the indefinite integrals of the f_k are uniformly absolutely continuous.