Department of Mathematics, National Cheng Kung University Oct. 2022

Qualifying Exam: Algebra

Answer all 7 questions, including all sub-parts. Show all your work and properly justify your answers.

Notations: G: a group, R: a ring with unity, $\mathbb{Z}$: the set of integers, $\mathbb{Q}$: the field of rational numbers.

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Problem 1: [Easy]. (10pts) Suppose G is simple and $|G| = 168$. How many elements of order 7 are there in G?

Problem 2: [Easy]. (10pts) Show that the set of nilpotent elements in a commutative ring R forms an ideal of R. (Recall $r \in R$ is nilpotent if $r^n = 0$ for some $n > 0$ in $\mathbb{Z}$)

Problem 3: [Easy]. (10pts) Prove that an Artinian integral domain is a field.

Problem 4: [Mid].(10pts) Determine all of the intermediate fields of the splitting field over $\mathbb{Q}$ of the polynomial $x^4 - 5x + 6 \in \mathbb{Q}[x]$.

Problem 5: [Mid]. Let $Z(G)$ denote the center of G.

a. (15pts) Prove that if $G/Z(G)$ is cyclic, then G is abelian.

b. (10pts) Suppose G is a non-abelian group of order p^3 for some prime p. Show that $|Z(G)| = p$.

Problem 6: [Mid]. (15pts) Suppose R is commutative. Let J be an ideal of R. Prove that for any R-module M, we have the following R-module isomorphism:

$$(R/J) \otimes_R M \cong M/JM$$

Problem 7: [Hard]. (20pts) Describe all prime ideals of the integral domain $\mathbb{Z}[x]$.

1 (10)	2 (10)	3 (10)	4 (10)	5 (25)	6 (15)	7(20)	Total (100)