Ph.D Qualify Exam **Analysis** **2022 Spring**

E: Easy; M: Moderate; D: Difficult

1(E, 20%, 2019, Fall). Let $k(x,y)$ be a measurable function on $\mathbb{R}^n \times \mathbb{R}^n$ satisfying that

$$\int_{\mathbb{R}^n} |k(x,y)|dy \le C \text{ for a.e. } x \text{ and } \int_{\mathbb{R}^n} |k(x,y)|dx \le C \text{ for a.e. } y,$$

where $C > 0$ is a universal constant. Prove that

$$(Tf)(x) := \int_{\mathbb{R}^n} k(x,y)f(y)dy$$

is a bounded operator on $L^p(\mathbb{R}^n)$ with $\|Tf\|_p \le C\|f\|_p$ for $1 \le p \le \infty$.

2(M, 20%, 2008, Spring). Prove that there exists an orthonormal basis for the subspace

$$\mathcal{B} = \left\{ f \in L^2([0,1]) \mid \int_0^1 \frac{|f(x)|}{x} dx < \infty \text{ and } \int_0^1 \frac{f(x)}{x} dx = 0 \right\}$$

of $L^2([0,1])$.

Hint: (i) Consider $\mathcal{F} = \left\{ f \in L^2([0,1]) \mid \int_0^1 \frac{|f(x)|}{x} dx < \infty \right\}$ and let T be an operator defined on $\mathcal{F}$ by $Tf = \int_0^1 \frac{f(x)}{x} dx$ for each $f \in \mathcal{F}$. (ii) Consider, for each $n \in \mathbb{N}$, the characteristic function $g_n = \chi_{[1/n,1]}$ of $[1/n, 1]$.

3(D, 20%, 2010, Spring). Let $1 < p < \infty$ and $f \in L^p(0,\infty)$. Define

$$F(x) = \frac{1}{x} \int_0^x f(t)dt, \quad 0 < x < \infty.$$

(a) Prove that $\|F\|_p \le \frac{p}{p-1}\|f\|_p$.

(b) Prove that the equality holds only if $f = 0$ a.e..

4(E, 20%, 2012, Spring). Show that the sequence

$$f_n(x) = \frac{1}{n} \sum_{k=1}^{n} \sin^2(kx)$$

converges in measure in $(-\pi, \pi)$.

5(M, 10%, 2014, Spring). Let $f \in L^1(\mathbb{R}^k)$. The maximal function $Mf(x)$ is defined as

$$Mf(x) = \sup_Q \frac{1}{|Q|} \int_Q |f(y)|dy,$$

where the sup is taken over all cubes Q with center x. Assume that both f and its maximal function Mf are in $L^1(\mathbb{R}^k)$. Prove that $f(x) = 0$ a.e..

6(M, 10%, 2015, Fall). Let $a = \{a_k\}_{k=1}^{\infty} \in \ell^p$ for some $1 < p < \infty$. Prove that

$$\lim_{p\to\infty} \|a\|_p = \|a\|_\infty.$$