Ph.D Qualify Exam **Analysis** **2021 Fall**

E: Easy; M: Moderate; D: Difficult

1(E, 20%, 2018, Spring). Let f_k,$f : \mathbb{R} \to \mathbb{R}$ be Lebesgue measurable functions and f_k converges to f in L^p, where $1 < p < \infty$. Prove that f_k converges to f in measure as $k \to \infty$.

2(M, 10%, 2018, Spring). Suppose that $f_k \to f$ in $L^3(\mathbb{R}^n)$, $g_k \to g$ a.e., and there exists $M > 0$ such that $\|g_k\|_{L^\infty(\mathbb{R}^n)} < M$ for all k. Prove that $f_k g_k \to fg$ in $L^3(\mathbb{R}^n)$.

3(E, 20%, 2021, Spring). Let $f \in L^1([0,\infty))$ and $a > 0$. Show that

$$\int_0^\infty \int_0^\infty \sin(ax) f(y) e^{-xy}\ dy\ dx = a \int_0^\infty \frac{f(y)}{a^2 + y^2}\ dy.$$

4(E, 10%, 2018, Fall). Let $\{f_k\}_{k\in\mathbb{N}}$ and f be Lebesgue measurable functions on a measurable set $E \subset \mathbb{R}^n$, where the Lebesgue measure of E is finite. Suppose that

$$\int_E \frac{|f_k(x) - f(x)|}{1 + |f_k(x) - f(x)|} dx \to 0$$

when $k \to \infty$. Prove or disprove that $f_k \to f$ in measure.

5(M, 20%, 2020, Fall). Suppose $f_k,\ f \in L^1(\mathbb{R}^n)$ and $f_k \to f$ a.e. Prove or disprove that $\int_{\mathbb{R}^n} |f_k(x)| dx \to \int_{\mathbb{R}^n} |f(x)| dx$ implies $\int_{\mathbb{R}^n} |f_k(x) - f(x)| dx \to 0$.

6(E, 10%). For $d \in \mathbb{N}$, let $B_d = \{(x_1, x_2, ..., x_d) \mid x_1^2 + x_2^2 + \cdots + x_d^2 \leq 1\}$. Drive the volume of B_d based on the knowledge about the gamma function $\Gamma(\frac{d}{2}) = \int_0^\infty r^{\frac{d}{2}-1} e^{-r}\ dr$ and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

7(E, 10%). Find the value of the integral

$$\int_0^\infty \frac{\sin x}{x} e^{-x} dx.$$