General Algebra

2021 Fall

Qualify Examination

October 8, 2021

Work out all of the eight problems. Show correct and clear arguments to get full credits.

1. (15 points) ** Suppose that G is an nonabelian group of order p^3. Show that the center of G is is of order p.
2. (10 points) *** Let p and q be distinct primes. Show that no groups of order p^2q are simple.
3. (10 points) *** Let G be a group. We set $G^{(1)} = G'$, the derived subgroup of G. For $i > 1$, define $G^{(i)} = (G^{(i-1)})'$. Show that each $G^{(i)}$, $i \geq 1$, is normal in G.
4. (15 points) * Describe all ring homomorphisms from $\mathbb{Z} \oplus \mathbb{Z}$ to $\mathbb{Z}$.
5. (10 points) * Let R be a finite ring with $|R| \geq 2$. If for $a, b \in R$, $ab = 0$ implies $a = 0$ or $b = 0$. Show that R is a division ring.
6. (15 points) ** Let R be a commutative ring and suppose that the subring A of R is contained in a finite union of prime ideals $P_1 \cup P_2 \cup \ldots \cup P_n$. Show that $A \subseteq P_i$ for some $1 \leq i \leq n$.
7. (15 points) ** Let $f(x) = x^3 - 6x^2 + 9x + 3 \in \mathbb{Q}[x]$, and let $K = \mathbb{Q}(u)$, where u is the real root of $f(x)$. Show that $\{1, u, u^2\}$ is a basis of K over Q, and express elements $u^4, (u+1)^{-1}$ in terms of this basis.
8. (10 points) *** Let K be a field with $|K| = q$, and let $f \in K[x]$ be irreducible. Prove that f divides $x^{q^n} - x$ if and only if $\deg(f)$ divides n.

Total number of points: 100