Numerical Analysis, Qualifying Exam. 2022-03

1. (15 pts) Estimate minimum N subdivisions of $[-1,1]$ so that the error of the numerical quadrature for

$$I = \int_{-1}^{1} \frac{1}{x+2} dx$$

is less than 10^{-8}, provided the composite Simpson's rule is adopted. (Hint: You may need to apply the theorem as below.)

Theorem *Suppose the integral $I = \int_a^b f(x)\,dx$ is estimated by Simpson's rule S_N using N subdivisions of $[a,b]$ and suppose that $f^{(4)}$ is continuous. Then the error in this approximation is given by*

$$I - S_N = -\frac{(b-a)\,h^4}{180} f^{(4)}(\xi)$$

where $h = (b-a)/N$ for some $\xi \in (a,b)$.

2. (10 pts) Consider the linear system $Ax = y$, where $A = \begin{bmatrix} a & -b \\ -c & d \end{bmatrix} \in \mathbb{R}^{2\times 2}$.

Give a sufficient condition so that both Jacobi iteration and Gauss-Seidel iteration are convergent. In addition, explain why the rate of convergence of Gauss-Seidel method is about twice faster than that of Jacobi method provided $bc \neq 0$.

3. The following gives an algorithm for the **inverse power iteration with variant shifts (INV-Shift)**:

i) *Give initial u_0 with ℓ-th component $u_0(\ell) = 1$, and the initial shift σ_0 nearby the target simple eigenvalue λ_* of $A \in \mathbb{R}^{n\times n}$. Let $k = 0$.*

ii) *Until convergence,* ***Do***

ii-1) solve the linear system $(A - \sigma_k I)v_k = u_k$

ii-2) Let $\alpha_k = e_\ell^T v_k = v_k(\ell) \neq 0$, the ℓ-th component of v_k.

ii-3) Set $\sigma_{k+1} = \sigma_k + \frac{1}{\alpha_k}$ and $u_{k+1} = \frac{v_k}{\alpha_k}$.

ii-4) $k = k + 1$.

(a) (10 pts) Suppose that INV-Shift converges, show that $\lim_{k\to\infty} \sigma_k = \lambda_*$ and u_k converges to the eigenvector of A corresponding to λ_* as $k \to \infty$

(b) (10 pts) Show that INV-Shift is equivalent to Newton's iteration for solving

$$F(x,\lambda) = \begin{bmatrix} (A - \lambda I)x \\ e_\ell^T x - 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Hence, you may conclude that INV-Shift converges locally qudratically.

4. It is well known that for any $a, b \in \mathbb{R}$,

$$fl(a \circ b) = (a \circ b)(1 + \delta), \qquad |\delta| \le \varepsilon_M,$$

where ε_M denotes the machine precision and '$\circ$' is an elementary arithmetic operator (i.e., $+, -, \times, \div$) and $fl(c)$ means that takes the floating-point number for the real number c.

(a) (10 pts) Show that if $|\delta_i| \le \varepsilon_M$ and $n\varepsilon_M \le 0.01$, then

$$1 - n\varepsilon_M \le \prod_{i=1}^{n}(1 + \delta_i) \le 1 + 1.01n\varepsilon_M.$$

(b) (5 pts) Let $x, y \in \mathbb{R}^n$. Show that

$$|fl(x^T y) - x^T y| \le \sum_{i=1}^{n} |\delta_i||x_i y_i| \le 1.01n\varepsilon_M \sum_{i=1}^{n} |x_i y_i|.$$

5. (10 pts) Let $x_0, x_1, \cdots, x_m$ be $m + 1$ distinct real numbers and $y_0, y_1, \cdots, y_m$ be $m + 1$ corresponding values. Show that there is an unique real-valued polynomial $p_n(x) = a_0 + a_1 x + \cdots + a_n x^n$ of degree $n \le m$ such that

$$\sum_{k=0}^{m} |p_n(x_k) - y_k|^2 = \min_{p(x)\in P_n[x]} \sum_{k=0}^{m} |p(x_k) - y_k|^2,$$

where $P_n[x]$ denotes the set of all real-valued polynomials of degree $\le n$.

6. (15 pts) Consider the model problem

$$u''(x) = f(x) \qquad \text{for } 0 < x < 1,$$
$$u(0) = u(1) = 0,$$

and the resulting linear system $AU = F$ by replacing u'' by the center finite difference formula

$$u''(x_i) \approx \frac{U_{i-1} - 2U_i + U_{i+1}}{h^2}$$

and $f(x_i)$ by F_i on a uniform mesh with size $h = \frac{1}{m+1}$. Find the eigenvalues and eigenvectors of A, and calculate the L^2-norm $||A||_2$ and the condition number $\kappa_2(A)$ of A.

7. (15 pts)

Let V_h be a finite-dimensional subspace of a Hilbert space V. Let $\{\phi_1, \ldots, \phi_M\}$ be a basis for V_h. Consider the discrete variaional problem (V_h): find $u_h \in V_h$ such that

$$a(u_h, v) = L(v), \quad \text{for all } v \in V_h.$$

where $a(.,.)$ is a symmetic bilinear form on $V \times V$ and L is a linear form on V. Derive the resulting linear system $AU = F$ for the problem (V_h) and show that the matrix A is positive definite provided $a(\cdot,\cdot)$ is V-elliptic, i.e. there exits a constant $\alpha > 0$ such that $\alpha||v|| \le a(v, v)$ for all $v \in V$.